Digital shapes, digital boundaries and rigid transformations: A topological discussion

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Abstract. Curvature is a continuous and infinitesimal notion. These properties induce geometrical difficulties in digital frameworks, and the following question is naturally asked: "How to define and compute curvatures of digital shapes?" In fact, not only geometrical but also topological difficulties are also induced in digital frameworks. The – deeper – question thus arises: "Can we still define and compute curvatures?" This latter question, that is relevant in the context of digitization, *i.e.*, when passing from $\mathbb{R}^n$ to $\mathbb{Z}^n$, can also be stated in $\mathbb{Z}^n$ itself, when applying geometric transformations on digital shapes. This paper proposes a preliminary discussion on this topic.

1. Introduction

In the continuous domain, the computation of curvature requires that the considered shapes – and more precisely their boundaries – present certain good properties, mostly in terms of differentiability. When passing from the continuous universe ($\mathbb{R}^n$) to the discrete one ($\mathbb{Z}^n$), the handling of curvature becomes much more complex. It is easy to guess that the induced difficulties derive from the necessity to model infinitesimal properties – namely the differentiability of boundaries – into a finite framework.

However, even before considering such geometrical concerns, it is crucial to keep in mind that there also exist topological concerns. Indeed, beyond its putative differentiability, the notion of boundary itself often becomes ill-defined in $\mathbb{Z}^n$. In other words, while the boundary of a continuous shape in $\mathbb{R}^n$ is an object of dimension $n-1$, and most often a $(n-1)$-manifold, it is unfortunately infrequent that the boundary of a digital shape in $\mathbb{Z}^n$ be a discrete hypersurface, and a fortiori a discrete $(n-1)$-manifold.

During the last decades, some efforts were devoted to tackle this issue in the context of digitization. More precisely, some conditions were provided to guarantee the preservation of good geometrical and topological properties of shape boundaries, when passing from $\mathbb{R}^n$ to $\mathbb{Z}^n$. However, if we now know how to correctly handle curvature during this digitization step, it remains challenging to also define adequate conditions for curvature definition and analysis when processing digital shapes. In particular, it is difficult to preserve correct topological – and thus geometrical – properties of digital shape boundaries when applying geometric transformations, even the most simple one such as rigid transformations.

In this paper – that is mainly related to the works published in [**?**, **?**] – we expose some preliminary results devoted to this question. More precisely, we focus on the specific case of digital shapes defined on $\mathbb{Z}^2$, and on their behaviour under rigid transformations. Considering such a low dimension and such simple transformations may seem meaningless and irrelevant at a time when the hot topics are related to high-dimension objects under arbitrary deformations. Nevertheless, beyond this apparent triviality, we show that the induced issues are not straightforward, and we

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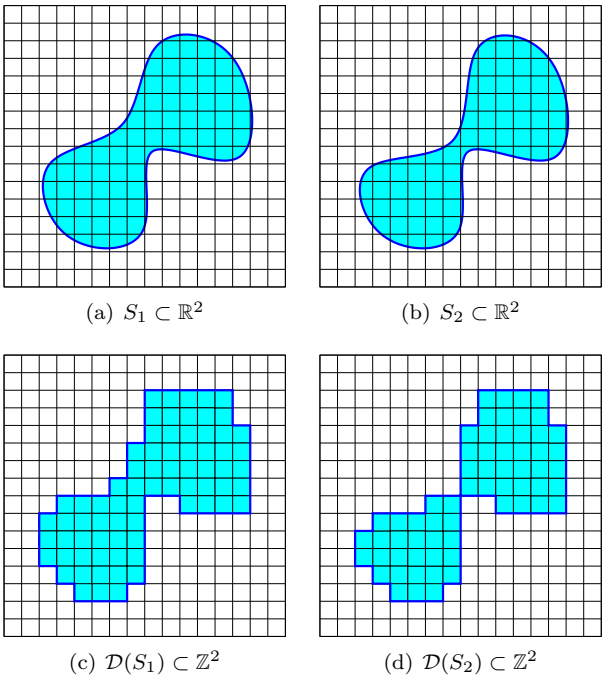


(a) $S_1 \subset \mathbb{R}^2$ (b) $S_2 \subset \mathbb{R}^2$

(c) $\mathcal{D}(S_1) \subset \mathbb{Z}^2$ (d) $\mathcal{D}(S_2) \subset \mathbb{Z}^2$

FIGURE 1. (a,b) Continuous shapes S_1 and S_2 in $\mathbb{R}^2$ (in cyan) and their boundaries (in blue). (c,d) The associated digital shapes $\mathcal{D}(S_1)$ and $\mathcal{D}(S_2)$ in $\mathbb{Z}^2$ (in cyan), and their digital boundaries (in blue). (c) The digital boundary of $\mathcal{D}(S_1)$ is a 1-manifold. (d) The digital boundary of $\mathcal{D}(S_2)$ is not a 1-manifold, by contrast with that of S_2.

intend to develop sound foundations for further developments at higher dimensions and for more general transformations.

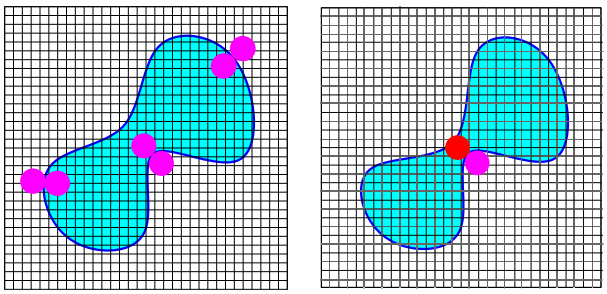
2. Digital shapes and their boundaries

Let us consider a finite closed set S in $\mathbb{R}^2$ whose boundary is a (set of) 1-manifold(s) as an original shape. Since computers handle only finite structures, such a continuous shape S is represented as a digital image, *i.e.*, a finite set of pixels associated to points of $\mathbb{Z}^2$. The induced digital shape is denoted by $\mathcal{D}(S)$, referring to the digitization procedure $\mathcal{D}$ that allows us to pass from $\mathbb{R}^2$ to $\mathbb{Z}^2$.

There exist several models for $\mathcal{D}$ [**?**]. For instance, if we consider the Gaussian model, we obtain $\mathcal{D}(S) = S \cap \mathbb{Z}^2$; in other words, the digital shape $\mathcal{D}(S)$ of S is simply obtained by "sampling" S with respect to the regular structure of $\mathbb{Z}^2$. We will note $\overline{\mathcal{D}(S)}$ the complement of $\mathcal{D}(S)$ in $\mathbb{Z}^2$.

It is mandatory to provide an explicit and sound definition for the notion of boundary of a digital shape $\mathcal{D}(S)$. To this end, let us first consider the links that exist between the points of $\mathbb{Z}^2$ and the pixels of a digital image. A pixel P, associated to a point $\mathbf{x}$ of $\mathbb{Z}^2$, can be seen as a unit square of $\mathbb{R}^2$ centered on $\mathbf{x}$. In other words, we have $P = \mathbf{x} + [-1/2, 1/2]^2 \subset \mathbb{R}^2$. From a structural point of view, the pixels of a digital image are nothing but the Voronoi cells of $\mathbb{R}^2$ induced by the points of $\mathbb{Z}^2$. In particular, some couples of pixels share a part of their (continuous) boundaries. More precisely, two pixels P_1 and P_2 associated to $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively, satisfy this assertion iff their exists an edge between $\mathbf{x}_1$ and $\mathbf{x}_2$ in the (dual) Delaunay diagram associated to the above Voronoi diagram.

Based on these considerations, the boundary $\partial\mathcal{D}(S)$ of the digital shape $\mathcal{D}(S)$, also called the digital boundary for short, is straightforwardly associated to the continuous boundary induced by the pixels of $\mathcal{D}(S)$. Indeed, $\partial\mathcal{D}(S)$ can be modeled by a set of couples of points $(\mathbf{x}, \overline{\mathbf{x}}) \in \mathcal{D}(S) \times \overline{\mathcal{D}(S)}$, that share an edge in the Delaunay diagram of $\mathbb{Z}^2$.



(a) (b)

FIGURE 2. (a) An r-regular shape (in cyan). Some inner and outer r-radius open balls are depicted in magenta, with r higher that the pixel size of the "topology-preserving" digitization grid. (b) A shape that is not r-regular (in cyan). An example of r-radius open balls couple is depicted, where the "inner" ball (in red) does not entirely lie into the shape.

The handling of digital boundaries can be considered from a topological point of view. To this end, we can use the standard notion of neighbourhood stated in digital topology [**?**]. The k-neighbourhood of a point $\mathbf{x} \in \mathbb{Z}^2$ is defined by $N_k(\mathbf{x}) = \{\mathbf{y} \in \mathbb{Z}^2 \mid \|\mathbf{x} - \mathbf{y}\|_p = 1\}$, for $k = 4, 8$ where $p = 1, \infty$, respectively. Then, the boundary of $\mathcal{D}(S)$ is defined by $\partial\mathcal{D}(S) = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x} \in \mathcal{D}(S), \mathbf{y} \in \overline{\mathcal{D}(S)}, \mathbf{y} \in N_4(\mathbf{x})\}$.

Figure 1 illustrates two examples of digitization procedure, where it is easily seen that the topology of a digital shape is not always the same as that of the initial continuous shape. More precisely, we observe that digital shape boundaries are not always guaranteed to be 1-manifolds, even though the original shape boundaries are.

3. Digitization and topology preservation

The issue of topological alteration of shape boundaries during the digitization process has been considered in the literature. In particular, Latecki *et al.* defined some conditions for guaranteeing boundary integrity, based on two key notions, namely r-regularity and well-composedness.

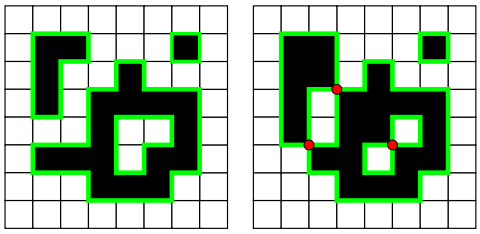
Definition 1 (r-regularity [**?**])**.** A closed set $S \subset \mathbb{R}^2$ is *r-regular* if for each boundary point of S, there exist two tangent open balls of radius r, lying entirely in S and its complement $\overline{S}$, respectively.

This definition derives from classical concepts of differential geometry, namely osculating balls and normal vectors. By considering the class of r-regular shapes in $\mathbb{R}^2$, the condition for preserving topological properties – especially in terms of boundaries – between a continuous shape and its digital counterpart is the following.

Proposition 2 ([**?**])**.** *An r-regular set $S \subset \mathbb{R}^2$ has the same topological structure as its digitized version $\mathcal{D}(S)$, for pixels of size $d < r$.*

This result is indeed an extension of the compatibility property between S and $\mathcal{D}(S)$, presented by Pavlidis in [**?**]. In particular, Latecki *et al.* were driven by more pragmatic motivations, related to some sampling devices for image acquisition, like CCD cameras. Figure 2 provides an example and a counterexample of r-regular shapes.

In [**?**], it was also shown that the (topology-preserving) digitization process of an r-regular shape must yield a well-composed shape [**?**], whose definition relies on the following concepts of adjacency, connectedness and connected components, in digital topology [**?**]. Let X be a digital shape in $\mathbb{Z}^2$. We say that two distinct points $\mathbf{x}, \mathbf{y}$ of X are k-adjacent (for $k = 4, 8$) if $\mathbf{x} \in N_k(\mathbf{y})$ (and – equivalently – $\mathbf{y} \in N_k(\mathbf{x})$). From the induced (symmetric) k-adjacency relation on X, we obtain, by reflexive-transitive closure, the (equivalence) k-connectedness relation. The k-connected components of X are the equivalence classes for this relation. From these notions, we can define the notion of well-composedness.



(a) (b)

FIGURE 3. (a) A well-composed shape X_1 of $\mathbb{Z}^2$ (in black). Its boundary ∂X_1 (in green) is a 1-manifold. (b) A shape X_2 of $\mathbb{Z}^2$ (in black) that is not well-composed. Its boundary ∂X_2 (in green) is not a 1-manifold (see the red dots).

Definition 3 (Well-composedness [**?**])**.** A digital shape X in $\mathbb{Z}^2$ is well-composed if each 8-connected component of X and of its complement $\overline{X}$ is also a 4-connected component.

Based on this definition, it is plain that the boundary ∂X of a digital shape X is a 1-manifold whenever X is well-composed. Some examples and counter-examples of well-composed shapes are provided in Figure 3.

As stated above, there actually exists a strong link between r-regularity and well-composedness.

Theorem 4 ([**?**])**.** *If S is r-regular, then $\mathcal{D}(S)$ is well-composed.*

Consequently, a continuous shape S of $\mathbb{R}^2$ that is r-regular – and whose boundary is a continuous 1-manifold that authorises curvature analysis – still presents good properties after digitization, since its digital counterpart $\mathcal{D}(S)$ is well-composed and then also presents as border $\partial\mathcal{D}(S)$ a 1-manifold.

4. Rigid transformations of digital shapes

As stated above, in the digital framework, curvature analysis makes sense when considering shapes that are well-composed. In the continuity of this result, the question that we consider now is the following: "What are the conditions for allowing curvature analysis not only on a digital shape X but also on its image by a geometric transformation?". In particular, we focus on the most simple – yet non-trivial – case of rigid transformations.

In $\mathbb{R}^2$, a rigid transformation is a function

$$\left| \begin{array}{ccccc} \mathcal{T} & : & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & & \mathbf{x} & \mapsto & \mathbf{R}.\mathbf{x} + \mathbf{t} \end{array} \right. \tag{4.1}$$

where $\mathbf{R}$ is a rotation matrix, *i.e.*, an element of the group $SO(2)$, and $\mathbf{t} \in \mathbb{R}^2$ is a translation vector. The rigid transformation $\mathcal{T}$ is a bijection, and its inverse function $\mathcal{T}^{-1}$ is also a rigid transformation.

Based on these definitions, the digital rigid transformations consist of composing the continuous rigid transformations with the standard rounding operator. We note $\mathcal{RIG}_{\mathbb{Z}^2}$ the set of all the digital rigid transformations.

Two transformation models can be considered for a digital rigid transformation T associated to a transformation $\mathcal{T}$. The Lagrangian (forwards) model consists of computing the direct image of the digital shape X by the transformation. However, as T is often neither injective nor surjective, this leads to topological defects. The Eulerian (backwards) model consists of computing the transformed image X_T as the shape whose image by the digital analogue T^{-1} of the inverse function $\mathcal{T}^{-1}$ of $\mathcal{T}$, lies into X. This is more satisfactory, since $\mathcal{T}^{-1}$ is defined on the whole transformed space $\mathbb{Z}^2$. In this model, that we consider hereafter, we have

$$X_T = D(\mathcal{T}(X \oplus \square)) \tag{4.2}$$

where $\oplus$ is the classical dilation operator defined in mathematical morphology [**?**, Ch. 1], and $\square$ is the unit square of $\mathbb{R}^2$, centered on the origin. These relationships with well-known concepts of mathematical morphology are actually not a coincidence, as it will be discussed later.

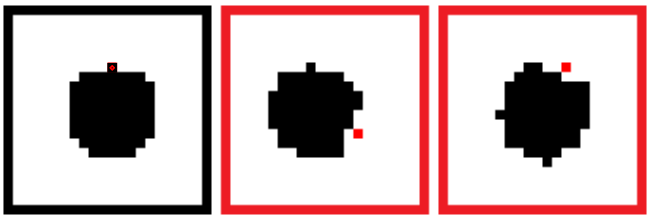


FIGURE 4. An original digital circle (left) and its rigid transformations (center and right). The original digital shape is well-composed while the transformed ones are not well-composed.

FIGURE 5. Forbidden patterns in regular shapes (up to $\pi/2$ rotations, symmetries and value inversion).

Unfortunately, the Eulerian model is not exempt from topological difficulties. In particular, the family of well-composed shapes is not stable with respect to $\mathcal{RIG}_{\mathbb{Z}^2}$. In other words, the transformed shape X_T obtained from a well-composed shape X with respect to a digital rigid transformation T is not necessarily well-composed itself. In that case, its topological properties are, of course, altered, and in particular, its boundary is no longer a (set of) 1-manifold(s). See Figure 4 for an example.

5. Rigid transformations and topology preservation

In this section, we summarise the main contribution of this work, that consists of defining a subfamily of well-composed shapes that remain stable – and topologically invariant – under any digital rigid transformations. The digital shapes forming this family are called *regular*, in reference to the above notion of r-regularity for continuous shapes in $\mathbb{R}^2$. The set of regular digital shapes can be defined in $\mathbb{Z}^2$ as follows.

Definition 5 ((Non-)singular shapes)**.** Let $X \subset \mathbb{Z}^2$ be a digital shape. We say that X is *singular* if at least one point $\mathbf{x}$ of X (resp. $\overline{X}$) has its whole 4-adjacent set included in $\overline{X}$ (resp. X).

Definition 6 (Regularity [**?**, **?**])**.** Let $X \subset \mathbb{Z}^2$ be a non-singular, well-composed shape. We say that X is *regular* if for any 4-adjacent points $\mathbf{x}, \mathbf{y} \in X$ (resp. $\overline{X}$), there exists a 2×2 set $\boxplus = \{z, z+1\} \times \{t, t+1\} \subset \mathbb{Z}^2$ such that $\mathbf{x}, \mathbf{y} \in \boxplus \subseteq X$ (resp. $\overline{X}$).

The regularity of a digital shape can be characterised as follows, and thus leads to a linear-time complexity pattern-based regularity analysis.

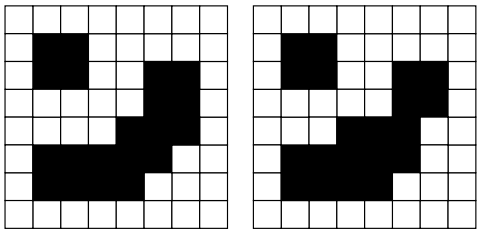
Proposition 7 ([**?**, **?**])**.** *A digital shape $X \subset \mathbb{Z}^2$ is regular iff none of the configurations depicted in Figure 5 appears in X and $\overline{X}$.*

Note that the first configuration (Figure 5(a)) characterises the non-well-composed shapes [**?**].

The main interest of regularity is to guarantee the stability of well-composed shapes – and their topological invariance – under any rigid transformation. In particular, we have the following result.

Theorem 8 ([**?**])**.** *Let $X \subset \mathbb{Z}^2$ be a well-composed shape. If X is regular, then, for any digital rigid transformation T, the transformed digital shape X_T is still well-composed, and has the same topological structure as X.*

Remark 9. This result establishes regularity as a sufficient condition for the stability of well-composedness, together with topological invariance. Our conjecture is however that this condition is also necessary.



(a) (b)

Figure 6. (a) A regular shape (in black). (b) A non-regular shape (in black) that is however opened and closed by a structuring element $\boxplus$.

Following mathematical morphology terminology [**?**, Ch. 1], if X is regular, then X is opened and closed by any structuring element $\boxplus$, *i.e.*

$$\gamma_{\boxplus}(X) = X \ominus \boxplus \oplus \boxplus = X \tag{5.1}$$

$$\phi_{\boxplus}(X) = X \oplus \boxplus \ominus \boxplus = X \tag{5.2}$$

However, the converse is not true, as illustrated in Figure 6. Nevertheless, it is plain that there exist strong links between these morphological operations and the notions of regularity and topological invariance. Our conjecture is that the regular shapes X are exactly those whose both the dilated $X \oplus \boxplus$ and the eroded $X \ominus \boxplus$ have the same topological structure as X. This intuition derives from the continuous analogue of this assertion for r-regular shape S of $\mathbb{R}^2$, where the discs of radius r play the role of $\boxplus$ [**?**].

6. Conclusion

This work constitutes a preliminary attempt to provide solutions for curvature definition and analysis of digital shapes, not only in their initial space but also in the spaces obtained under rigid transformations.

Some results have been proposed in the specific case of shapes in $\mathbb{Z}^2$ under rigid transformations. It has been proved in [**?**] that these results can, of course, be interpreted in the framework of binary digital images, but also extended to grey-level and label images, thus providing efficient image processing and analysis strategies.

Nevertheless, many efforts remain to do towards solutions in more general cases. In particular, the handling of (i) higher dimensions, *i.e.*, $\mathbb{Z}^3$ and more generally $\mathbb{Z}^n$ for $n \geq 3$, and (ii) arbitrary geometric transformations, still remain open issues. Moreover, it may be important to derive not only sufficient, but also necessary conditions, for curvature analysis. It may additionally be useful to deal with both Eulerian and Lagrangian models.

Even though this preliminary study allows us to understand and solve some topological problems of digital shapes under their rigid transformations, geometrical problems still remain: geometries of digital shapes are not preserved under rigid transformations in general. For example, see Figure 7. It is expected to investigate geometry-preserving conditions of digital shapes during their rigid transformations as well.

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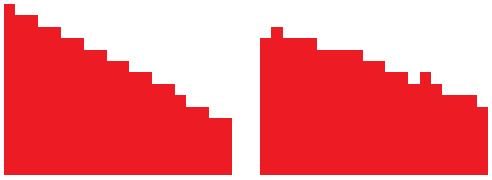

(a) (b)

FIGURE 7. (a) A digital half plane, which is regular, and (b) its transformed shape, which is still well-composed (*i.e.*, the topology is preserved) but not a digital half plane any more (*i.e.*, the geometry is not preserved).

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