On 2-D simple sets in n-D cubic grids

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Abstract Preserving topological properties of objects during reduction procedures is an important issue in the field of discrete image analysis. Such procedures are generally based on the notion of *simple point*, the exclusive use of which may result in the appearance of "topological artifacts". This limitation leads to consider a more general category of objects, the *simple sets*, which also enable topology-preserving image reduction. A study of 2-dimensional simple sets in 2-dimensional spaces has been proposed recently. This article is devoted to the study of 2-dimensional simple sets in spaces of higher dimension (*i.e.* n-dimensional spaces, $n \geq 3$). In particular, several properties of *minimal* simple sets (*i.e.* which do not strictly include any other simple sets) are proposed, leading to a characterisation theorem. It is also proved that the removal of a 2-dimensional simple set from an object can be performed by only considering the *minimal* ones, thus authorising the development of efficient thinning algorithms.

Keywords Digital topology · Thinning · Topology preservation · Simple sets · Cubical complexes · n-dimensional spaces

1 Introduction

The preservation of topological properties is fundamental in many applications of image analysis. Consequently, an intensive effort has been conducted for more than 40 years [1, 2] to develop topology-preserving methods enabling to process discrete binary images, essentially to perform skeletonisation, homotopic reduction, or segmentation. Most of these methods are based on the well known notion of *simple point* [3].

Since the definition of the *deletable elements* of a 2-dimensional binary image by Rosenfeld [4] in 1970, efforts have been directed towards local characterisations of simple points in 2-, 3- and 4-dimensional spaces [5–8] together with more accurate definitions of simplicity.

Some (less frequent) studies have also been devoted to the more general notion of *simple sets*. Given a discrete binary object X, a simple set is a part $S \subset X$ which - similarly to simple

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points - can be removed from X, providing a result $X \setminus S$ having the same topology (*i.e.* the same homotopy type) as X. The study of simple sets has, in particular, been motivated by the existence of sets composed only of simple points but which are not simple, as pointed out by Rosenfeld [4].

Based on this consideration, the only simple sets studied for a long time have been those composed only of (parallely [9] or sequentially [10, 11]) simple points. The exclusive focus on such simple sets can be explained by the fact that Ronse proved [12] in 1986 that in a 2-dimensional digital image (*i.e.* an image defined on $\mathbb{Z}^2$, equipped with a $(4, 8)$- or $(8, 4)$-adjacency framework), any simple set can be removed by *successive* removal of simple points. Afterwards, remained for a long time the hope of extending this theorem to higher dimensions [13]. Unfortunately, it was recently proved [14] that there exist simple sets in 3-dimensional spaces composed only of non-simple points, emphasising the fact that the exclusive use of the notion of simple point is insufficient to enable the development of robust topology-preserving reduction procedures.

This unexpected result has led to start new studies [15, 16] on simple sets without simple points, in the framework of cubical complexes [17] (which enables to retrieve the main concepts of digital topology in $\mathbb{Z}^n$, but also to model a larger class of discrete cubical objects). In this context, an extension of Ronse's theorem to 2-dimensional pseudomanifolds has already been established [16].

In this article, we present a complete study of 2-dimensional simple sets in n-dimensional cubical spaces ($n \geq 3$), motivated in particular by the fact that Ronse's theorem cannot be generalised to spaces of such dimensions (as it will be shown in Section 3). We prove several properties on 2-dimensional *minimal simple sets* (*i.e.* simple sets which do not strictly include any other simple sets) and we finally establish a characterisation of these objects. We also prove that the detection and removal of a 2-dimensional simple set from an object can be performed by only considering the *minimal ones*, thus authorising to further develop new efficient thinning algorithms.

The sequel of this article is organised as follows. Section 2 recalls background notions related to cubical complexes and simple sets. Sections 3 to 5 contain the main contribution of this article. They present new results related to 2-dimensional simple sets in n-dimensional spaces, in particular necessary properties of 2-dimensional minimal simple sets (Section 3), a characterisation of such sets (Section 4), and a proof of non-deterministic decomposition of 2-dimensional simple sets into a sequence of minimal ones (Section 5). Section 6 summarises the contributions of the article and describes further works. Proofs of the propositions and theorems can be found in Appendix B (some of these proofs are based on results established in previous works and auxiliary propositions which are provided in Appendix A). A more technical discussion on a connectedness property of 2-dimensional minimal simple sets related to the dimension of their embedding space is proposed in Appendix C.

2 Cubical complexes and simple sets

This section provides the minimal set of background notions required to make this paper self-included. More details can be found *e.g.* in [8, 15].

2.1 Basic notions

Let $\mathbb{Z}$ be the set of integers. Let $\mathbb{F}_0^1 = \{\{a\} \mid a \in \mathbb{Z}\}$ and $\mathbb{F}_1^1 = \{\{a, a+1\} \mid a \in \mathbb{Z}\}$. Let $n \geq 1$.

Let $f \subset \mathbb{Z}^n$. If f is the Cartesian product of m elements of $\mathbb{F}^1_1$ and $n - m$ elements of $\mathbb{F}^1_0$, we say that f is a *face* or an *m-face* (*of* $\mathbb{Z}^n$), m is the *dimension* of f, and we write $\dim(f) = m$. We denote by $\mathbb{F}^n$ the set composed of all faces of $\mathbb{Z}^n$.

Let $f \in \mathbb{F}^n$ be a face. We set $\hat{f} = \{g \in \mathbb{F}^n \mid g \subseteq f\}$, and $\hat{f}^* = \hat{f} \setminus \{f\}$ (see Fig. 1). Any $g \in \hat{f}$ is a *face of* f (or *of* $\hat{f}$).

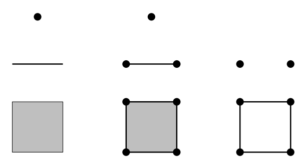

Fig. 1 Left: three faces f_i ($i \in [0, 2]$) of dimension i, respectively. Middle: $\hat{f}_i$. Right: $\hat{f}_i^*$. In black (dots and lines): 0 and 1-faces, respectively; in grey: 2-faces.

Let $F \subset \mathbb{F}^n$ be a finite set of faces. We set $F^- = \bigcup_{f \in F} \hat{f}$ (see Fig. 2).

Let $F \subset \mathbb{F}^n$ be a set of faces. Let $f \in F$ be a face. We set $star(f, F) = \{g \in F \mid f \subseteq g\}$ and $star^*(f, F) = star(f, F) \setminus \{f\}$ (see Fig. 2).

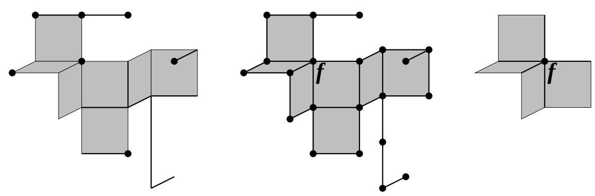

Fig. 2 Left: a set of faces F. Middle: the closure F^- of F. Right: $star(f, F^-)$, for the 0-face $f \in F^-$.

Let $F \subset \mathbb{F}^n$ be a finite set of faces. We say that F is a *cell* or an *m-cell* if there exists an m-face $f \in F$ such that $F = \hat{f}$. We say that F is a (*cubical*) *complex* (*in* $\mathbb{F}^n$) if for any $f \in F$, we have $\hat{f} \subseteq F$, *i.e.*, if $F = F^-$. If F is a complex in $\mathbb{F}^n$, we write $F \preceq \mathbb{F}^n$.

Let $F \preceq \mathbb{F}^n$ be a complex. Any subset $G \subseteq F$ which is also a complex is a *subcomplex* of F (see Fig. 3). If G is a subcomplex of F, we write $G \preceq F$. If $G \preceq F$ and $G \neq F$, we write $G \prec F$.

Let $F \subset \mathbb{F}^n$ be a finite set of faces. Let $f \in F$ be a face. The face f is a *facet* of F if there is no $g \in F$ such that $f \in \hat{g}^*$. We denote by F^+ the set composed of all facets of F (see Fig. 3).

Let $F \preceq \mathbb{F}^n$ be a complex. Let $G \preceq F$ be a subcomplex of F. If $G^+ \subseteq F^+$, we say that G is a *principal subcomplex of* F, and we write $G \sqsubseteq F$ (see Fig. 4). If $G \sqsubseteq F$ and $G \neq F$, we write $G \sqsubset F$.

Let $F \preceq \mathbb{F}^n$ be a complex such that $F \neq \emptyset$. The *dimension* of F is defined by $\dim(F) = \max\{\dim(f) \mid f \in F^+\}$. We say that F is an *m-complex* if $\dim(F) = m$. We say that F is a *pure complex* if for all $f \in F^+$, we have $\dim(f) = \dim(F)$.

Definition 1 Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \preceq F$ be a subcomplex of F. We set $F \oslash G = (F^+ \setminus G^+)^-$. The set $F \oslash G$ is a complex which is the *detachment of* G *from* F (see Fig. 4).

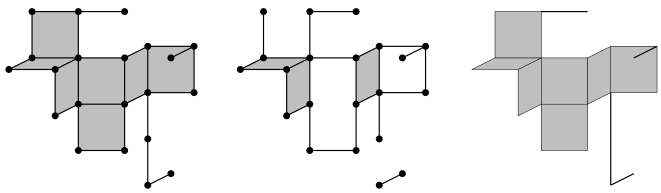

Fig. 3 Left: a complex F. Middle: a subcomplex G of F ($G \preceq F$). Right: the set $F^+ \subseteq F$ of the facets of F.

Definition 2 Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \preceq F$ be a subcomplex of F. The *attachment of* G *to* F is the complex defined by $Att(G, F) = G \cap (F \oslash G)$ (see Fig. 4).

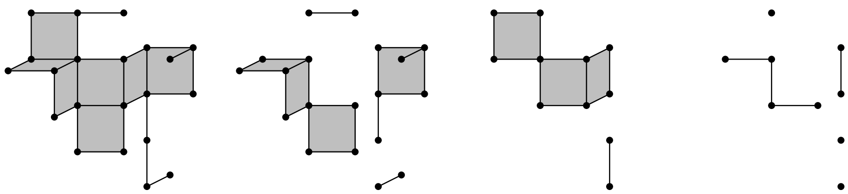

Fig. 4 From left to right: a complex F, a principal subcomplex G of F ($G \sqsubseteq F$), the detachment of G from F, the attachment of G to F.

2.2 Topology in cubical complexes

Let $F \subset \mathbb{F}^n$ be a finite set of faces such that $F \neq \emptyset$. A sequence of faces $(f_i)_{i=0}^s$ ($s \geq 0$) is a *path in* F *(from* f_0 *to* f_s*)* if for all $i \in [0, s-1]$, either f_i is a face of f_{i+1} or f_{i+1} is a face of f_i (with $f_i, f_{i+1} \in F$). If $k = \min\{\dim(f_i) \mid i \in [0, s]\}$, we also say that $(f_i)_{i=0}^s$ is a k*-path* (*in* F). We say that F is k*-connected* if, for any facets $f, g \in F^+$, there is a k-path in F from f to g. Let $G \subseteq F$ be a subset of F. We say that G is a k*-connected component of* F if G is k-connected and is maximal for this property (*i.e.*, we have $H = G$ whenever $G \subseteq H \subseteq F$ and H is k-connected). Note that, in particular, a k-connected component G of a complex F is a principal subcomplex of F, *i.e.* $G \sqsubseteq F$. We denote by $\mathcal{C}_k[F]$ the set of all k-connected components of F. We say that F is *connected* (resp. *strongly connected*) if F is 0-connected (resp. 1-connected), and we note $\mathcal{C}[F]$ for $\mathcal{C}_0[F]$. The number of distinct connected components of F is denoted by $|\mathcal{C}[F]|$ (more generally, the notation $|X|$ will be used to denote the cardinal of any finite set X).

Collapsing is a topological operation on complexes that preserves homotopy type.

Definition 3 Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $f \in F$. If $g \in \hat{f}^*$ is such that f is the only face of F which strictly includes g, then we say that g is a *free face for* F, that f is a *border face for* F, and that the pair (f, g) is a *free pair for* F (see Fig. 5). If (f, g) is a free pair for F, we say that the complex $F \setminus \{f, g\}$ is an *elementary collapse of* F.

Property 1 Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. If (f, g) is a free pair for F, then $f \in F^+$ and $\dim(f) = \dim(g) + 1$.

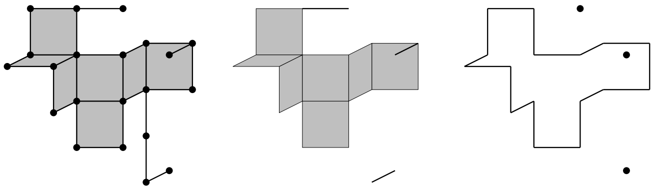

Fig. 5 Left: a complex F. Middle: the border faces for F. Right: the free faces for F. Any pair (f, g) such that f is in the middle subfigure and $g \subset f$ is in the right subfigure is a free pair for F.

Definition 4 Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \preceq F$ be a subcomplex of F. We say that F *collapses onto* G, and we note $F \searrow G$, if there exists a sequence of complexes $\langle F_i \rangle_{i=0}^t$ ($t \geq 0$) such that $F_0 = F$, $F_t = G$, and F_i is an elementary collapse of F_{i-1} for all $i \in [1, t]$ (in particular, if G is a 0-cell, we say that F is *collapsible*). The sequence $\langle F_i \rangle_{i=0}^t$ is a *collapse sequence from F to G* (see Fig. 6). We will also call collapse sequence from F to G, the sequence $\langle (f_i, g_i) \rangle_{i=1}^t$ of free pairs verifying $F_i = F_{i-1} \setminus \{f_i, g_i\}$ for all $i \in [1, t]$.

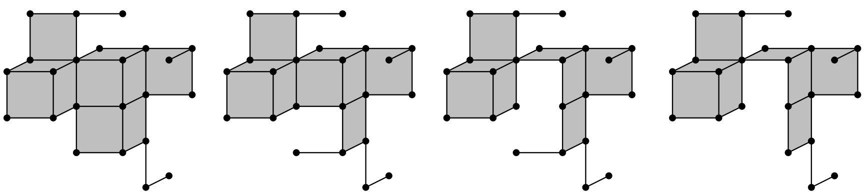

Fig. 6 From left to right: a collapse sequence from F to a subcomplex $G \preceq F$.

Let $F \preceq \mathbb{F}^n$ be a cubical complex. The *Euler characteristic* of F, written $\chi(F)$, is defined by $\chi(F) = \sum_{i=0}^{n} (-1)^i . \nu_i$, where ν_i is the number of i-faces of F for $i \in [0, n]$. From Property 1, it is easy to check that the collapse operation preserves the Euler characteristic. Moreover, it may be easily proved that the connectedness is also preserved by this operation (see *e.g.* [16]).

Property 2 Let $n \geq 1$. Let $F, G \preceq \mathbb{F}^n$ be cubical complexes. If $F \searrow G$, then:

- $\chi(F) = \chi(G)$;
- $|\mathcal{C}[F]| = |\mathcal{C}[G]|$.

2.3 Simple sets

Intuitively, a cell $G \preceq F$ is called simple if there is a topology-preserving deformation of the complex F over itself onto the relative complement of G in F. The following definition of simple cells, based on the collapse operation, can be seen as a discrete counterpart of the one given by Kong in [6].

Definition 5 Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $f \in F$ be a face of F. The cell $\hat{f}$ is a simple cell for F if $F \searrow F \oslash \hat{f}$.

Proposition 1 ([6]) *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $f \in F^+$ be a facet of F such that* $\dim(f) = 2$. *Then $\hat{f}$ is a simple cell for F if and only if $C[Att(\hat{f}, F)] = 1$ and $\chi(Att(\hat{f}, F)) = 1$.*

Definition 5, proposed for simple cells, naturally extends to subcomplexes $G \preceq F$ which contain an arbitrary number of facets, leading to the notion of *simple set*.

Definition 6 Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \preceq F$ be a subcomplex of F. We say that G is *simple for F* if $F \searrow F \oslash G$. Such a subcomplex G is called a *simple set for F* (or a *k-dimensional simple set for F*, if $\dim(G) = k$).

The notion of attachment leads to the following local characterisation of simple sets.

Proposition 2 ([18]) *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \preceq F$ be a subcomplex of F. The complex G is simple for F if and only if $G \searrow Att(G, F)$.*

Remark 1 Without loss of generality, the study of the simple sets G of a complex F can be restricted to those verifying $(\emptyset \sqsubset)\ G \sqsubset F$ (see [15] for a full justification). From now on, we will always implicitly consider that a simple set verifies these properties.

We introduce now the notion of *minimal simple sets*, which correspond to simple sets that do not strictly include any other simple sets.

Definition 7 Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubseteq F$ be a principal subcomplex of F. The complex G is a *(k-dimensional) minimal simple set for F* if G is a (k-dimensional) simple set for F and G is minimal (w.r.t. $\sqsubseteq$) for this property (*i.e.* $\forall H \sqsubseteq G$, H is simple for $F \Rightarrow H = G$).

The notion of minimal simple set may be useful from both theoretical and algorithmic points of view since (*i*) the existence of a simple set necessarily implies the existence of at least one minimal simple set, and (*ii*) by definition, a minimal simple set is necessarily easier (or, at least, not harder) to characterise than a "general" simple set. In particular, we can hope that in several cases (depending on the dimension(s) of $\mathbb{F}^n$ and/or of F, for instance), the study of minimal simple sets could be sufficient to deal with the problem of detaching *any* simple set from a complex.

3 Properties of 2-D minimal simple sets

3.1 Existence

In [16], it has been proved that in $\mathbb{F}^2$, all the minimal simple sets (and in particular those of dimension 2) are cells.

Proposition 3 ([16]) *Let $F \preceq \mathbb{F}^2$ be a cubical complex. Let $G \sqsubset F$ be a minimal simple set for F. Then $|G^+| = 1$,* i.e. *G is a simple cell for F.*

However, this property is no longer true in $\mathbb{F}^n$ ($n \geq 3$), as stated by the following proposition.

Proposition 4 *Let $n \geq 3$. There exist $F \preceq \mathbb{F}^n$ and $G \sqsubset F$ such that* $\dim(G) = 2$, $|G^+| > 1$ *and G is a minimal simple set for F.*

In order to prove this proposition, it is sufficient to build a complex $F \preceq \mathbb{F}^n$ ($n \geq 3$) and a subcomplex $G \sqsubset F$ of dimension 2 such that G is a minimal simple set for F while $|G^+| > 1$. An example of such complexes F and G is provided in Fig. 7. Note that since a complex $F \preceq \mathbb{F}^3$ can be embedded in $\mathbb{F}^n$ for any $n > 3$, the proof can be restricted to the case $n = 3$.

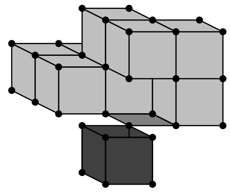

Fig. 7 A (pure) 2-dimensional complex $F \preceq \mathbb{F}^3$ including a minimal simple set G which is not a cell (in medium and light grey). See Subsection B.1 for a full description.

3.2 Basic properties

3.2.1 Connectedness

In [15], it has been proved that a minimal simple set is necessarily connected. The following property of 2-dimensional minimal simple sets is then a specific case of this result.

Proposition 5 ([15]) *Let $n \geq 3$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubset F$ be a 2-dimensional minimal simple set for F. Then $|\mathcal{C}[G]| = 1$, i.e. G is connected.*

From Prop. 2 and Property 2, we immediately derive the following result.

Proposition 6 *Let $n \geq 3$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubset F$ be a 2-dimensional minimal simple set for F. Then $|\mathcal{C}[Att(G,F)]| = 1$, i.e. $Att(G,F)$ is connected.*

Remark 2 For the same reasons, we also have $\chi(Att(G,F)) = \chi(G)$.

3.2.2 Purity

In [15], it has been proved: (i) that a (minimal) simple set cannot contain any 0-facet, and (ii) that a minimal simple set contains a 1-facet if and only if it is a simple 1-cell. The following proposition is a straightforward consequence of these results.

Proposition 7 *Let $n \geq 3$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubset F$ be a 2-dimensional minimal simple set for F. Then, $\forall f \in F^+, \dim(f) = 2$, i.e. G is pure.*

3.2.3 Strong connectedness

The following proposition states that a 2-dimensional minimal simple set is not only connected but also *strongly* connected.

Proposition 8 *Let $n \geq 3$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubset F$ be a 2-dimensional minimal simple set for F. Then, $|\mathcal{C}_1[G]| = 1$, i.e. G is strongly connected.*

3.3 Initial facet

In this subsection, we focus on the first facets removed during the collapse sequences enabling to detach 2-dimensional minimal simple sets *which are not simple cells*. Such facets, called *initial facets* present several important properties, especially related to the attachment of the 2-dimensional minimal simple sets.

Definition 8 Let $n \geq 3$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubset F$ be a 2-dimensional minimal simple set for F such that $|G^+| \geq 2$. We say that $f \in G^+$ is an *initial facet of* G (*in* F) if there exists a collapse sequence $\langle (f_i, g_i) \rangle_{i=1}^t$ ($t \geq 1$) from F to $F \oslash G$ such that $f = f_1$.

3.3.1 Possible configurations

Note that, from Prop. 7, an initial facet is necessarily a 2-face. Moreover, as stated by the following proposition, an initial facet can only present very few configurations w.r.t. its attachment.

Proposition 9 *Let* $n \geq 3$. *Let* $F \preceq \mathbb{F}^n$ *be a cubical complex. Let* $G \sqsubset F$ *be a* 2*-dimensional minimal simple set for* F *such that* $|G^+| \geq 2$. *Let* $f \in G^+$ *be an initial facet of* G. *Then* $\dim(Att(\hat{f}, F)) = 1$ *and* $|C[Att(\hat{f}, F)]| = 2$.

Remark 3 The attachment $Att(\hat{f}, F)$ of the cell $\hat{f}$ generated by an initial facet f of G necessarily corresponds to one of the three configurations (up to rotations and symmetries) illustrated in Fig. 8. The reader may easily check that these three configurations are the only ones presenting the required properties.

3.3.2 Uniqueness

The following proposition states that the attachment of a 2-dimensional minimal simple set G is a cell, all the faces of which are included in a same unique facet of G.

Proposition 10 *Let* $n \geq 3$. *Let* $F \preceq \mathbb{F}^n$ *be a cubical complex. Let* $G \sqsubset F$ *be a* 2*-dimensional minimal simple set for* F *such that* $|G^+| \geq 2$. *Let* $f \in G^+$ *be an initial facet of* G. *Then:*

(i) $\exists g \in F, Att(G, F) = \hat{g} \wedge \hat{g} \in C[Att(\hat{f}, F)]$,
i.e. $Att(G, F)$ *is a cell which is a connected component of* $Att(\hat{f}, F)$*;*
(ii) $\forall k \in G^+, k \neq f \Rightarrow \hat{k} \cap Att(G, F) = \emptyset$,
i.e. f *is the only facet of* G *including faces of* $Att(G, F)$.

From this proposition, we straightforwardly derive the uniqueness of the initial facet of a 2-dimensional minimal simple set.

Proposition 11 *Let* $n \geq 3$. *Let* $F \preceq \mathbb{F}^n$ *be a cubical complex. Let* $G \sqsubset F$ *be a* 2*-dimensional minimal simple set for* F *such that* $|G^+| \geq 2$. *Then* G *has exactly one initial facet.*

Props. 10 and 11 motivate the following extension of Definition 8

Definition 9 Let $n \geq 3$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubset F$ be a 2-dimensional minimal simple set for F such that $|G^+| \geq 2$. In addition to Definition 8, we also say that f is *the* initial facet of G (in F) if f is the (only) facet of G such that $\hat{f}$ intersects (and actually includes) $Att(G, F)$. The cell $\hat{f} \sqsubset G$ is called *the initial cell* of G.

Remark 4 From Prop. 10, the initial facet is the only facet f of G including faces of $Att(G, F)$. Consequently, the attachment of the initial cell $\hat{f}$ corresponds to one of the three configurations (up to rotations and symmetries) illustrated in Fig. 9.

3.4 Composition

The following proposition describes the composition of a 2-dimensional minimal simple set, especially w.r.t. the properties of its facets.

Proposition 12 *Let $n \geq 3$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubset F$ be a* 2*-dimensional minimal simple set for F such that $|G^+| \geq 2$. Then G^+ is composed of:*

- 1 *facet f such that* $\dim(Att(\hat{f}, F)) = 1$ *and* $|\mathcal{C}[Att(\hat{f}, F)]| = 2$ *(*i.e. *the initial facet);*
- $|G^+| - 1$ *(> 0) facets g_i ($i \in [1, |G^+| - 1]$) such that $Att(\hat{g}_i, G) = \hat{g}_i^*$.*

In particular, except the initial facet, G contains no border faces for G (and a fortiori *for F).*

4 Characterisation of 2-D minimal simple sets

The properties stated in the previous section provide necessary conditions for defining a 2-dimensional set as being a minimal simple one. We provide hereafter a set of *necessary and sufficient* conditions, then enabling to characterise a 2-dimensional minimal simple set.

Theorem 1 *Let $n \geq 3$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubset F$ be a principal subcomplex of F such that* $\dim(G) = 2$ *and* $|G^+| \geq 2$*. Then, G is a minimal simple set for F if and only if all the following conditions hold:*

(i) $|\mathcal{C}[Att(G, F)]| = 1$,
i.e. *$Att(G, F)$ is connected;*

(ii) $\exists! f \in G^+, \hat{f}^* \neq Att(\hat{f}, F)$,
i.e. *G has exactly one border face in F;*

(iii) $\exists! f \in G^+, \hat{f}^* \cap Att(G, F) \neq \emptyset$,
i.e. *there exists exactly one facet of G that includes faces of $Att(G, F)$;*

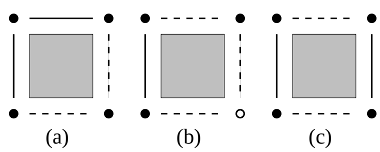

Fig. 8 The three possible configurations (up to rotations and symmetries) for the attachment $Att(\hat{f}, F)$ of the cell $\hat{f}$ generated by an initial facet f (in grey) of a 2-dimensional minimal simple set G for F. Full lines and disks: $Att(\hat{f}, F)$; dot lines and empty disks: $\hat{f}^* \setminus Att(\hat{f}, F)$. The 2-dimensional minimal simple set illustrated in Fig. 7 and described in Subsection B.1 has an initial facet (in medium grey) which corresponds to configuration (a). The other two configurations can be easily obtained from the same example by elementary rotations and translations of A^- and K^- (see definitions in Subsection B.1).

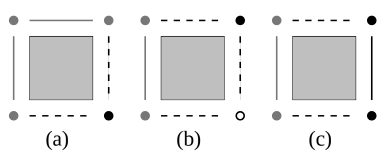

Fig. 9 The three possible configurations (up to rotations and symmetries) for the attachment of the initial cell $\hat{f}$ of a 2-dimensional minimal simple set G for F. In grey (full lines and disks): $Att(\hat{f}, G)$; in black (full lines and disks): $Att(G, F)$; in black (dot lines and empty disks): $\hat{f}^* \setminus Att(\hat{f}, F)$.

(iv) $\exists f \in G, \dim(f) = 0 \wedge G \searrow \hat{f}$,
i.e. *G is collapsible.*

Remark 5 Alternative versions of the previous characterisation can be obtained by substituting the following condition to condition (*i*):

(i') $Att(G, F)$ is a 0-cell or a 1-cell;

or the following three conditions to condition (*iv*):

(iv') G is connected;
(v') $\chi(G) = 1$;
(vi') $G \searrow H$ with $\dim(H) = 1$.

In order to illustrate the minimality of this characterisation, we provide in Fig. 10 some counter-examples (*i.e.* some *non-simple* or *non-minimal* simple sets) for which exactly one of the conditions of Theorem 1 is not verified.

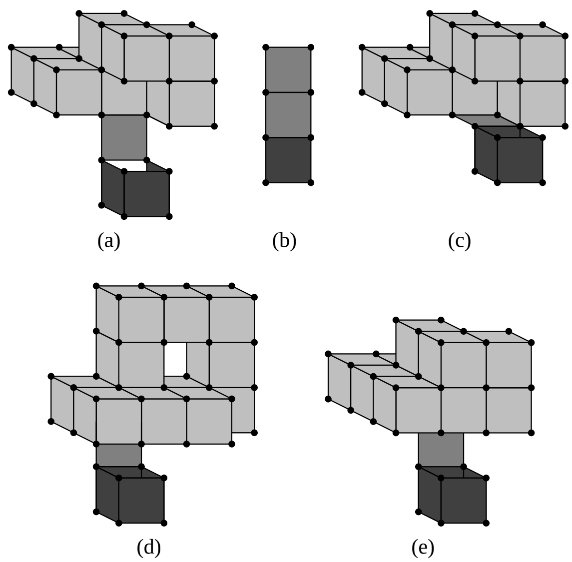

Fig. 10 Examples of subcomplexes $G \sqsubset F \preceq \mathbb{F}^3$ which do not satisfy all the conditions of Theorem 1. In light, medium and dark grey: F; in light and medium grey: G; in medium grey: border face(s) of G. (a) G verifies conditions (*ii*), (*iii*) and (*iv*) but not condition (*i*) ($Att(G, F)$ is not connected). As G is connected, $G \not\searrow Att(G, F)$: G is not simple. (b) G verifies conditions (*i*), (*iii*) and (*iv*) but not condition (*ii*) (G has two border faces). Obviously, G is not minimal. (c) G verifies conditions (*i*), (*ii*) and (*iv*) but not condition (*iii*) (4 principal cells of G intersect $Att(G, F)$). The medium grey cell is obviously a simple cell for F: G is simple but not minimal. (d) G verifies conditions (*i*), (*ii*) and (*iii*) but not condition (*iv*) (G is not collapsible since it has a tunnel). We have $0 = \chi(G) \neq \chi(Att(G, F)) = 1$, then $G \not\searrow Att(G, F)$: G is not simple. (e) G verifies conditions (*i*), (*ii*), (*iii*), (*iv*'), and (*v*') but not condition (*vi*'). In particular, the light-grey part of G is a dunce hat [19], which cannot be modified by any collapse operation. Then, $G \not\searrow Att(G, F)$: G is not simple. Note that in (a) and (c), the light grey part of F is equal to the complex L^- defined in Appendix B.1.

5 Confluence property of 2-D simple sets

The following theorem guarantees that any simple set of dimension lower or equal to 2 can be detached by iterative detachment of minimal simple sets. Note that this property does not depend on the order of the detachments. It can thus lead to the development of *non-deterministic* algorithms, justifying the importance of the study of (2-dimensional) minimal simple sets.

Theorem 2 *Let* $n \geq 1$*. Let* $F \preceq \mathbb{F}^n$ *be a cubical complex. Let* $G \sqsubset F$ *be a simple set for* F *such that* $\dim(G) \leq 2$*. If* G *is not a minimal simple set, then:*

(i) $\exists H \sqsubset G$ *such that* H *is a minimal simple set for* F*;*
(ii) $\forall H \sqsubset G$ *such that* H *is a minimal simple set for* F*,* $G \oslash H$ *is a simple set for* $F \oslash H$*.*

Remark 6 This theorem extends (i) the result proposed in [15] which states that any 1-dimensional simple set in $\mathbb{F}^n$ can be removed by iterative (and non-deterministic) detachment of simple 1-cells, and (ii) the result proposed in [16] which states that any simple set for a complex embedded in a 2-dimensional (pseudo)manifold in $\mathbb{F}^n$ can be removed by iterative (and non-deterministic) detachment of simple cells.

Remark 7 This proposition cannot be generalised to $\dim(G) = 3$ (and *a fortiori* $\dim(G) > 3$). In particular (i) is still obviously true (from the very definition of a minimal simple set), but (ii) is false, as illustrated by classical counter-examples such as the Bing's house [20] or less classical ones [15].

6 Conclusion

From Theorem 1, we are now able to characterise *all* the minimal simple sets of dimension 2, independently of the dimension of their embedding space. Moreover, from Theorem 2 we also know that *any* simple set of dimension 2 can be fully detached from a complex by iterative (and non-deterministic) detachment of minimal simple sets.

Based on these two results, it becomes possible to develop topology-preserving reduction algorithms providing principal subcomplexes G of complexes $F \preceq \mathbb{F}^n$ which no longer include any 1-dimensional and 2-dimensional simple sets.

From an applicative point of view, further works will now consist in proposing such efficient algorithms (which could be used, for instance, to optimise the computation of the homology of geometrically complex objects [21–23]). In particular, the main purpose will be to simultaneously reach the optimum time and space algorithmic complexities. From a theoretical point of view, the next step will be the study of (minimal) simple sets of higher dimensions (already initiated in [14]).

A Auxiliary properties

Proposition 13 **([15])** *Let* $n \geq 1$*. Let* $F \preceq \mathbb{F}^n$ *be a cubical complex. Let* $G \sqsubset F$ *be a simple set for* F*. Let* $H \sqsubset G$ *be a strict principal subcomplex of* G*. If* $Att(H, F) \subseteq Att(G, F)$*, then:*

(i) H *is a simple set for* F*;*
(ii) $G \oslash H$ *is a simple set for* $F \oslash H$*.*

Proposition 14 ([15]) *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubset F$ be a minimal simple set for F. Then, there exists at most one 0-face $f \in G$ such that $star^*(f, F)$ is not connected and $\mathcal{C}[star^*(f, F)] \cap \mathcal{C}[star^*(f, G)] \neq \emptyset$ (*i.e.* $star^*(f, F)$ has a connected component included in G). If f exists, then we have $Att(G, F) = \{f\}$, and $star^*(f, G)$ is connected.*

The following proposition is a particular case of a proposition established in [15].

Proposition 15 (From [15]) *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubset F$ be a simple set for F. Then, $\forall H \sqsubset G$ such that $Att(H, F)$ is a 0-cell:*

(i) H is a simple set for F;
(ii) $G \oslash H$ is a simple set for $F \oslash H$.

The proposition below is an easy consequence of a lemma established in [15] which proves, in particular, that a collapse operation on a complex F cannot "remove" a path in F from $f \in F$ to f, passing exactly once through a 1-facet of F.

Proposition 16 (From [15]) *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubset F$ be a simple set for F such that G contains a 1-facet $f \in F^+$. Let C_1, C_2 be the connected components of $G \setminus \{f\}$ containing respectively g_1, g_2, the 0-faces of f. Then $C_1 \neq C_2$ and for any connected subcomplex $H \preceq F \setminus \{f\}$, $H \cap C_1 = \emptyset$ or $H \cap C_2 = \emptyset$.*

Proposition 17 ([24]) *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \preceq F$ be a subcomplex of F. If $F \searrow G$ then there exists a* decreasing *collapse sequence $\langle (f_i, g_i) \rangle_{i=1}^t$ ($t \geq 1$) from F to G,* i.e. *a sequence such that $\forall i \in [1, t-1], dim(f_i) \geq dim(f_{i+1})$.*

Proposition 18 *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let a be a 0-face of F such that $F \searrow \{a\}$ (*i.e. *F is collapsible). Then, for all 0-face $b \in F$, $F \searrow \{b\}$.*

Proof If $\dim(F) = 0$, the result is obvious. If $\dim(F) \geq 1$ then, from Prop. 17, there exists a 1-complex G containing the same 0-faces as F, and such that $F \searrow G$ and $G \searrow \{a\}$. From Property 2, we have $\chi(G) = \chi(\{a\}) = 1$ and $|\mathcal{C}[G]| = |\mathcal{C}[\{a\}]| = 1$. It is known that a connected 1-complex G such that $\chi(G) = 1$ collapses onto any of its 0-faces. Hence the result holds. □

Proposition 19 *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let (f, g) be a free pair for F such that* $\dim(f) \leq 2$. *Let $\langle (f_i, g_i) \rangle_{i=1}^t$ ($t \geq 1$) be a collapse sequence of F such that $f, g \in \{f_i, g_i\}_{i=1}^t$. Then, there exists a collapse sequence $\langle (f'_i, g'_i) \rangle_{i=1}^t$ of F such that $(f'_1, g'_1) = (f, g)$ and $\{f_i, g_i\}_{i=1}^t = \{f'_i, g'_i\}_{i=1}^t$.*

Proof Let $\alpha \in [1, t]$ such that $f_\alpha = f$ (α exists as $f \in F^+$ and $f \in \{f_i, g_i\}_{i=1}^t$).
Case 1: $g_\alpha = g$. As the pair (f_α, g_α) is free for F, by moving it at the beginning of the collapse sequence, we are obviously done.
Case 2: $g_\alpha \neq g$. Let $\beta \in [1, t]$ such that $g \in \{f_\beta, g_\beta\}$. Note that since (f, g) is a free pair for F, we necessarily have $\alpha < \beta$ and $g = f_\beta$. Thus, from Property 1, $\dim(f) = 2$ (otherwise $\dim(f) = 1$ and $\dim(g_\beta) = \dim(g) - 1 = \dim(f) - 2 < 0$). Let $F_1 = F \setminus \{f_i, g_i\}_{i=1}^{\alpha}$, $G = (\{f_i, g_i\}_{i=1}^t)^-$ and $H = \{f_i, g_i\}_{i=\alpha+1}^t$. Note that by construction, $G \sqsubset F$ is simple for F, and $H^- \sqsubset F_1$ is simple for F_1. Since collapse operations in distinct connected components of a complex are fully independent, we can suppose without loss of generality that G is connected and, by Property 2, that $Att(G, F)$ is connected. Let h_1, h_2 be the two 0-faces of $\hat{g}$. Let C_1, C_2 be the connected components of $H^- \setminus \{g\}$ containing h_1 and h_2, respectively. By Prop. 16, we know that $C_1 \neq C_2$ and it follows from the connectedness of $Att(G, F)$ that $Att(G, F) \cap C_1 = \emptyset$ or $Att(G, F) \cap C_2 = \emptyset$. Without loss of generality, we can consider that $Att(G, F) \cap C_2 = \emptyset$. Therefore $(C_2 \cup \hat{g}) \sqsubseteq F_1$ and the connected component of $\hat{f}^* \setminus \{g, g_\alpha\}$ containing h_2 is included in C_2. Let a_2 be the 0-face of $\hat{g}_\alpha$ belonging to C_2. From Prop. 15, we deduce that $C_2 \cup \hat{g}$ is simple for F_1, and thus collapses onto $\hat{h_1}$, while C_1 collapses onto $Att(H^-, F_1)$. Then, C_2 collapses onto $\hat{h_2}$. Moreover, from Prop. 18, C_2 collapses onto $\hat{a_2}$. Hence, we can remove $\{f_\alpha, g_\alpha\} \cup H$ from $F_1 \cup \{f_\alpha, g_\alpha\}$ with a sequence of elementary collapses by removing (f, g), $C_2 \setminus \{a_2\}$, (g_α, a_2) and $C_1 \setminus Att(H^-, F_1)$ in this order. The result then follows from Case 1. □

B Proofs

B.1 Proof of Proposition 4

The proof is proposed for $n = 3$. It is *a fortiori* true for $n > 3$. The proposed complex is the one illustrated in Fig. 7. Let $F \preceq \mathbb{F}^3$ be the (pure) 2-dimensional complex generated by the set of principal faces $F^+ = K \cup A \cup L$, with:

K (in dark grey in Fig. 7) $= \{\{3\}\times\{0,1\}\times\{-1,0\}, \{4\}\times\{0,1\}\times\{-1,0\}, \{3,4\}\times\{0\}\times\{-1,0\}, \{3,4\}\times\{1\}\times\{-1,0\}\}$;

A (in medium grey in Fig. 7) $= \{\{2,3\}\times\{1,2\}\times\{0\}\}$;

L (in light grey in Fig. 7) $= \{\{0\}\times\{0,1\}\times\{0,1\}, \{0\}\times\{1,2\}\times\{0,1\}, \{0\}\times\{2,3\}\times\{0,1\}, \{1\}\times\{1,2\}\times\{0,1\}, \{1\}\times\{1,2\}\times\{1,2\}, \{2\}\times\{0,1\}\times\{0,1\}, \{2\}\times\{1,2\}\times\{0,1\}, \{2\}\times\{2,3\}\times\{1,2\}, \{3\}\times\{1,2\}\times\{1,2\}, \{3\}\times\{2,3\}\times\{0,1\}, \{3\}\times\{2,3\}\times\{1,2\}, \{0,1\}\times\{0\}\times\{0,1\}, \{0,1\}\times\{3\}\times\{0,1\}, \{1,2\}\times\{0\}\times\{0,1\}, \{1,2\}\times\{1\}\times\{0,1\}, \{1,2\}\times\{1\}\times\{1,2\}, \{1,2\}\times\{2\}\times\{0,1\}, \{1,2\}\times\{2\}\times\{1,2\}, \{1,2\}\times\{3\}\times\{0,1\}, \{2,3\}\times\{1\}\times\{1,2\}, \{2,3\}\times\{2\}\times\{0,1\}, \{2,3\}\times\{3\}\times\{0,1\}, \{2,3\}\times\{3\}\times\{1,2\}, \{0,1\}\times\{0,1\}\times\{0\}, \{0,1\}\times\{0,1\}\times\{1\}, \{0,1\}\times\{1,2\}\times\{0\}, \{0,1\}\times\{1,2\}\times\{1\}, \{0,1\}\times\{2,3\}\times\{0\}, \{0,1\}\times\{2,3\}\times\{1\}, \{1,2\}\times\{0,1\}\times\{0\}, \{1,2\}\times\{0,1\}\times\{1\}, \{1,2\}\times\{1,2\}\times\{2\}, \{1,2\}\times\{2,3\}\times\{0\}, \{1,2\}\times\{2,3\}\times\{1\}, \{2,3\}\times\{1,2\}\times\{1\}, \{2,3\}\times\{1,2\}\times\{2\}, \{2,3\}\times\{2,3\}\times\{0\}, \{2,3\}\times\{2,3\}\times\{2\}\}$.

Let S be a collapse sequence from F to K^- (such a collapse sequence exists; some of its steps are provided in Fig. 11). Let $G = (F^+ \setminus K)^- = (A \cup L)^- \sqsubset F$. By construction, $F \searrow F \oslash G$, then G is a simple set for F. Let us suppose that G is not minimal. Then, there exists a subcomplex $H \sqsubset G$ such that H is simple for F. Since $H^+ \subset G^+$, we have $F \oslash G (= K^-) \sqsubset F \oslash H$. As F is connected and $F \searrow F \oslash H$, from Property 2, $F \oslash H$ is also connected. By construction, K^- is a connected component of $F \oslash A^- = (K \cup L)^-$. Then, we must have $A^- \sqsubseteq F \oslash H$ (otherwise, $K^- \sqsubset F \oslash H \sqsubseteq F \oslash A^-$: contradiction). Thus, $H \sqsubseteq L^-$. As F is pure, there does not exist any border 1-face in F. Moreover, F has no border face belonging to L (the obvious proof, by observation of F, is left to the reader). Then, there does not exist any free face for F in L^-, and *a fortiori* in H, which is then not simple: contradiction. Consequently, G is a minimal simple set for $F \preceq \mathbb{F}^3$ while $|G^+| = 39 \neq 1$. □

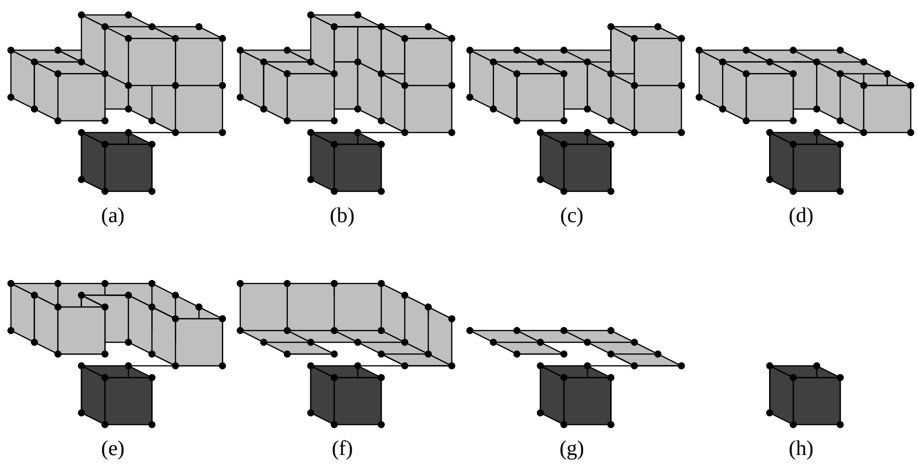

Fig. 11 From (a) to (h): some steps of a collapse sequence from F (Fig. 7) to K^- (h) (see text).

B.2 Proof of Proposition 8

The proof is obvious if $|G^+| = 1$. We now suppose that $|G^+| \geq 2$. Let us suppose that there exists $G_1, G_2 \sqsubset G$ such that $G = G_1 \cup G_2$ and $\dim(G_1 \cap G_2) = 0$.

– If $G_1 \cap G_2 \subseteq Att(G, F)$ then $Att(G_1, F) \subseteq Att(G, F)$, and from Prop. 13, G_1 is simple for F, in contradiction with the minimality of G.

– If $G_1 \cap G_2 \not\subseteq Att(G,F)$, then there exists a 0-face $f \in G_1 \cap G_2$ such that $f \notin Att(G,F)$. But, then, $star^*(f,F)$ is not connected and any connected component of $star^*(f,F)$ is also a connected component of $star^*(f,G)$. Thus, from Prop. 14, we have $Att(G,F) = \{f\}$: contradiction.

Consequently, there are no principal subcomplexes $G_1, G_2 \sqsubset G$ such that $G = G_1 \cup G_2$ and $\dim(G_1 \cap G_2) = 0$. It easily follows that G is strongly connected. □

B.3 Proof of Proposition 9

Since $\hat{f}$ is not a connected component of F (otherwise, $\hat{f}$ is also a connected component of G, in contradiction with Prop. 5), $Att(\hat{f},F)$ has at least one connected component. Moreover, since f is a border 2-face for F and $\hat{f}$ is not simple for F (otherwise G would not be minimal), $Att(\hat{f},F)$ has at least two connected components. But, from Prop. 14, $Att(\hat{f},F)$ has at most one connected component of dimension 0, which makes impossible the existence of more than two connected components for $Att(\hat{f},F)$. Then, $Att(\hat{f},F)$ has exactly two connected components, one of which is necessarily of dimension 1. □

B.4 Proof of Proposition 10

If $Att(\hat{f},F)$ has a 0-dimensional connected component (*i.e.* f corresponds to configurations (a) or (b) of Fig. 8) then (*i*) and (*ii*) directly follow from Prop. 14. We now suppose that $Att(\hat{f},F)$ has two 1-dimensional connected components (*i.e.* f corresponds to configuration (c) of Fig. 8). Let $\langle (f_i,g_i)\rangle_{i=1}^{t}$ ($t \geq 1$) be a collapse sequence from F to $F \oslash G$ such that $f_1 = f$. We set $F_1 = F \setminus \{f_1, g_1\}$ and $G_1 = (\{f_i,g_i\}_{i=2}^{t})^-$. Obviously, $G_1 \sqsubset F_1$ is simple for F_1. Let h be the only 1-face of $\hat{f} \setminus \{g_1\}$ free for F, h_1, h_2 be the two 0-faces of $\hat{h}$, and b_1 be the 1-face of $\hat{f} \setminus \{g_1, h\}$ including h_1. Let C_1, C_2 be the connected components of $G_1 \setminus \{h\}$ containing h_1 and h_2, respectively. As h is a facet of G_1, from Prop. 16, we have $C_1 \neq C_2$ and we can suppose without loss of generality that $Att(G,F) \cap C_2 = \emptyset$. Then, $C_2 \cup \hat{h} \sqsubset F_1$ and, from Prop. 15, we deduce that $C_2 \cup \hat{h}$ is simple for F_1. Consequently, $C_2 \cup \hat{f}$ is simple for F, and the minimality of G implies that $C_2 \cup \hat{f} = G$. As $C_1 \cap G_1^+ = \emptyset$ we have $Att(G,F) = \hat{b}_1$, then (*i*) holds. As $C_1 \neq C_2$, h_1 is not included in any face of $G^+ \setminus \{f\}$. From Prop. 19, by exchanging g_1 and h, the same can be said for the other 0-face of b_1, hence (*ii*) holds. □

B.5 Proof of Proposition 12

Let $f \in G^+$ be the initial facet of G. By definition, $|G^+| \geq 2$. Let us suppose that there exists a (2-)facet $g \in G^+$ ($g \neq f$) such that $Att(\hat{g}_i, G) \neq \hat{g}_i^*$. From Prop. 19, there exists a collapse sequence $S = \langle (f_i,g_i)\rangle_{i=1}^{t}$ ($t \geq 1$) from F to $F \oslash G$ such that $f_1 = g$, and g is then an initial facet of G, in contradiction with Prop. 11. Then, for all facets $g \in G^+$ such that $g \neq f$, we have $Att(\hat{g}_i, G) = \hat{g}_i^*$. □

B.6 Proof of Theorem 1 ("⇒" side)

We suppose that G is a minimal minimal simple set for F. Then, (*i*), (*ii*), and (*iii*) are proved by Props. 6, 12, and 10, respectively. Moreover, (*iv*) easily derives from Props. 2 and 10. □

B.7 Proof of Theorem 1 ("⇐" side)

We suppose that G satisfies conditions (*i*) to (*iv*).
Proof of simpleness.
Case 1: $\dim(Att(G,F)) = 0$. From (*i*), there exists a 0-face $g \in G$ such that $Att(G,F) = \hat{g}$. From (*iv*) and Prop. 18, $G \searrow \hat{g} = Att(G,F)$. Then, from Prop. 2, G is simple for F.
Case 2: $\dim(Att(G,F)) = 1$. From (*iii*) there exists exactly one facet of G that includes faces of $Att(G,F)$. Let $f \in G^+$ be this facet. Then, $Att(G,F) \preceq \hat{f}^*$, and as $\dim(Att(G,F)) = 1$, we have $\dim(\hat{f}) = 2$. Moreover, from (*iv*), we know that G is connected, and then we have $Att(G,F) \neq \hat{f}^*$ (otherwise, there would exist at least two facets of G that include faces of $Att(G,F)$). Then, from (*i*), we deduce that there exists two distinct 0-faces

$h_1, h_2 \in Att(G, F)$ which are free for $Att(G, F)$. Let $b_1, b_2 \in \hat{f} \setminus Att(G, F)$ be the 1-faces such that $h_1 \subset b_1$ and $h_2 \subset b_2$. Still from (*iii*), we have $Att(\hat{f}, G) \cap Att(\hat{f}, F \oslash G \cup \hat{f}) = \emptyset$, and then b_1, b_2 are necessarily free for F. From (*iv*) and Property 2, we deduce that G is connected, and then $Att(\hat{f}, G) \neq \emptyset$ (in particular, this implies that $b_1 \neq b_2$). If $\dim(Att(\hat{f}, G)) = 0$, we deduce from (*ii*) that no facet of G (except f) can be removed by a collapse sequence, in contradiction with (*iv*). Then, $\dim(Att(\hat{f}, G)) = 1$. Consequently the only possible configuration for $\hat{f}$ verifies $C[Att(\hat{f}, F)] = \{\hat{g}, \hat{h}\}$ where g, h are distinct 1-faces of $\hat{f}^*$ (note that this configuration will actually correspond to configuration (c) in Fig. 8). In particular, and without loss of generality, we can suppose that $h_1, h_2 \in \hat{g}$. Let $S = \langle (f_i, g_i) \rangle_{i=1}^{t}$ ($t \geq 1$) be a collapse sequence from G to $\{h_1\}$ (such a collapse sequence exists from (*iv*) and Prop. 18). From Prop. 19, we can suppose without loss of generality that $(f_1, g_1) = (f, b_1)$, and then we necessarily have $(f_t, g_t) = (g, h_2)$. In particular, $\langle (f_i, g_i) \rangle_{i=1}^{t-1}$ is then a collapse sequence from G to $\hat{g}$.
In every cases, $G \searrow \hat{g} = Att(G, F)$ and then G is simple for F.
Proof of minimality.
Let $H \sqsubseteq G$ be a principal subcomplex of G, simple for F. Let F_1 be the connected component of F including G. We have in particular $F_1 \searrow F_1 \oslash G$ and $F_1 \searrow F_1 \oslash H$. Then, from Property 2, $F_1 \oslash G$ and $F_1 \oslash H$ are connected. The complex H contains at least one border face f for F (otherwise, it could not be detached). Then $f \in H$, and $F_1 \oslash G \subseteq F_1 \oslash H \subseteq F_1 \oslash \hat{f}$. But, from (*ii*) and (*iii*), f is the only facet of G including faces of $F_1 \oslash G$. Thus, $F_1 \oslash G$ is a connected component of $F_1 \oslash \hat{f}$. Then, as $F_1 \oslash H$ is connected and $F_1 \oslash G \subseteq F_1 \oslash H$, we obtain $F_1 \oslash G = F_1 \oslash H$. Finally, since G and H are principal subcomplexes of F, we have $G = H$. □

B.8 Proof of Remark 5

Proof of (*i*), (*ii*), (*iii*), (*iv*) $\Leftrightarrow$ (*i'*), (*ii*), (*iii*), (*iv*). From Prop. 10, the attachment of a 2-dimensional minimal simple set is a 0-cell or a 1-cell. Conversely, if the attachment is a cell then it is connected.
Proof of (*iv*) $\Leftrightarrow$ (*iv'*), (*v'*), (*vi'*). If G is collapsible, then from Property 2, $\chi(G) = 1$ and G is connected, and from Prop. 17, it can be collapsed onto a 1-subcomplex H by using a decreasing collapse sequence from G to one of its 0-cells. Conversely, if G verifies $\chi(G) = 1$, G is connected and $G \searrow H$ with $\dim(H) = 1$ then, from Property 2, $\chi(H) = 1$ and H is connected, and it is then known that H is collapsible. It follows that G is also collapsible. □

B.9 Proof of Theorem 2

The proof of (*i*) easily follows from the definitions. Let $H \sqsubset G$ such that H is a minimal simple set for F. Let $S = \langle (f_i, g_i) \rangle_{i=1}^{t}$ ($t \geq 1$) be a collapse sequence from F to $F \oslash G$, and $S' = \langle (f'_i, g'_i) \rangle_{i=1}^{s}$ ($s \geq 1$) be a collapse sequence from F to $F \oslash H$. We obviously have $\{f'_i, g'_i\}_{i=1}^{s} \subset \{f_i, g_i\}_{i=1}^{t}$ (in particular, $s < t$). Hence, by induction from Prop. 19, we can build a new collapse sequence from F to $F \oslash G$, the first s free pairs of which are those of S'. It follows that $F \oslash H \searrow F \oslash G = (F \oslash H) \oslash (G \oslash H)$. Hence (*ii*) holds. □

C A strong connectedness property valid only in $\mathbb{F}^3$

As the attachment of the initial cell $\hat{f}$ of a 2-dimensional minimal simple set G to this simple set is connected, it easily follows that $G \oslash \hat{f}$ is also connected, but it may happen that $G \oslash \hat{f}$ looses the strong connectedness property of G. However, in $\mathbb{F}^3$, the space is not "large enough" to enable $G \oslash \hat{f}$ to become non-strongly connected. This property of strong connectedness preservation can present an algorithmic interest for the development of topology-preserving reduction procedures based on the detachment of (2-dimensional) minimal simple sets, and especially for their optimisation in dimension 3.

Proposition 20 *Let $F \preceq \mathbb{F}^3$ be a cubical complex. Let $G \sqsubset F$ be a 2-dimensional minimal simple set for F such that $|G^+| \geq 2$. Let $\hat{f} \sqsubset G$ be the initial cell of G. Then $G \oslash \hat{f}$ is strongly connected.*

Proof Let S be a collapse sequence from F to $F \oslash G$. Let (c_1, g_0) be the first free pair of S containing a facet of G distinct from the initial facet f (note that (c_1, g_0) is necessarily the second or the third free pair of S). Obviously, $g_0 \in Att(\hat{f}, G)$ and c_1 is the unique facet of $G \oslash \hat{f}$ including g_0. (The reader may refer to Fig. 12 to "visualise" the adjacency relations between the different faces considered in this proof.)

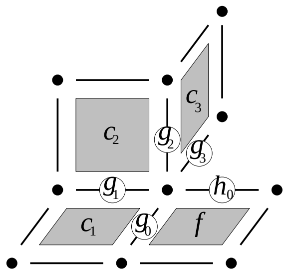

Fig. 12 Configuration of a 2-dimensional minimal simple set in the neighbourhood of its initial facet (see text).

- If $Att(\hat{f}, G) = \{g_0\}$, we set $E = \hat{c_1}$. We have $Att(\hat{f}, G) \prec E \sqsubseteq G \oslash \hat{f}$ and E is strongly connected.
- Otherwise, from Remark 3, $Att(\hat{f}, G) = \{g_0, h_0\}^-$ where g_0 and h_0 are two distinct 1-faces including a same 0-face. We set $s_0 = g_0 \cap h_0$. The cell $\hat{c_1}$ contains g_0, then it contains two 1-faces including s_0: g_0 and g_1 ($g_1 \notin \hat{f}$ as $\hat{c_1} \neq \hat{f}$). Since f is the only border face for G (Prop. 12), g_1 is not free in G. Then, there exists another 2-cell $\hat{c_2} \neq \hat{c_1}$ of G containing g_1 (and then different from $\hat{f}$). The cell $\hat{c_2}$ contains s_0 since $s_0 \subset g_1$, and a second 1-face $g_2 \neq g_1$ including s_0. We have $g_2 \neq g_0$ as $c_1 \neq c_2$.
  - If $g_2 = h_0$, then we set $E = \{c_1, c_2\}^-$. Obviously, E is strongly connected and $Att(\hat{f}, G) \prec E \sqsubseteq G \oslash \hat{f}$.
  - Otherwise, as g_2 is not a border face for G, there exists a cell $\hat{c_3} \neq \hat{c_2}$ of G containing g_2 (and then different from $\hat{f}$). The cell $\hat{c_3}$ contains g_2 and then s_0, and it follows that there exists a second 1-face $g_3 \in \hat{c_3}$ including s_0. We have $g_3 \neq g_0$, as g_0 is a free face for $G \oslash \hat{f}$, and $g_3 \neq g_1$, otherwise the cells $\hat{c_3}$ and $\hat{c_2}$ would share two 1-faces: $g_3 = g_1$ and g_2. The subset A of $star^*(s_0, G \oslash \hat{f})$ defined by $A = \{c_1, c_2, c_3, g_0, g_1, g_2, g_3\}$ is strongly connected. Symmetrically, we can define in $star^*(s_0, G \oslash \hat{f})$ a strongly connected subset B containing four distinct 1-faces (one of them being h_0). However, $star^*(s_0, \mathbb{F}^3)$ contains exactly six 1-faces and $A, B \subset star^*(s_0, \mathbb{F}^3)$. Consequently, $A \cap B \neq \emptyset$ and $E = (A \cup B)^-$ is strongly connected and verifies $Att(\hat{f}, G) \prec E \sqsubseteq G \oslash \hat{f}$.

In all the possible cases, we can build a complex E strongly connected and verifying $Att(\hat{f}, G) \prec E \sqsubseteq G \oslash \hat{f}$. Let a, b be two (2-)facets of $G \oslash \hat{f}$. As G is strongly connected, there exists a 1-path π from a to b in G. If f is an element of π, its predecessor and its successor in π are then faces of $Att(\hat{f}, G)$, and thus of E. Consequently, we can modify π in order to obtain a 1-path from a to b in $G \oslash \hat{f}$ (it is sufficient to replace in π all the occurrences of the 2-face f by a well-chosen 1-path in E). Then $G \oslash \hat{f}$ is strongly connected. □

Remark 8 This property is no longer true for $n = 4$ (and *a fortiori* $n > 4$). In $\mathbb{F}^4$, the complex $G = (\{f\} \cup B)^-$ described hereafter (and illustrated in Fig. 13) is a minimal simple set, the initial facet of which is f, and such that $B^- \subset \{(x, y, z, t) \mid x, y, z, t \geq 0\}$. Let C be the image of B by the central symmetry w.r.t. the point $(0, 0, 0, 0)$. The complex $(\{f\} \cup B \cup C)^-$ is a minimal simple set for a well-chosen complex F, and is no longer strongly connected after the detachment of its initial cell $\hat{f}$ ($\dim(B^- \cap C^-) = 0$).
$f = \{-1, 0\} \times \{0, 1\} \times \{0\} \times \{0\}$;

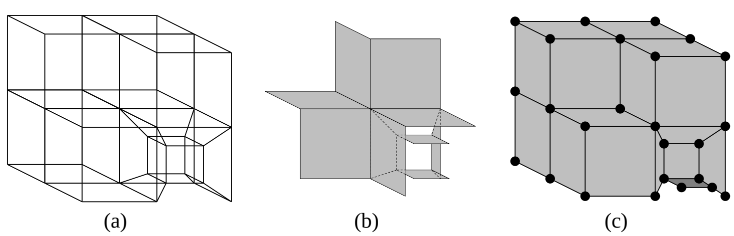

Fig. 13 Different views of the (pure) 2-dimensional minimal simple set $G = (\{f\} \cup B)^- \preceq \mathbb{F}^4$ described in Remark 8. (a) Visualisation of the 1-faces of B^-. (b) Visualisation of the "missing" 2-faces in G (*i.e* the 2-faces g such that $\hat{g}^* \preceq G$ while $g \notin G^+$). (c) Visualisation of G (B in light grey, f in dark grey).

$B = \{\{1\} \times \{0,1\} \times \{0\} \times \{0,1\}, \{0,1\} \times \{0,1\} \times \{0\} \times \{0\}, \{0,1\} \times \{0\} \times \{0,1\} \times \{0\}, \{0,1\} \times \{1\} \times \{0,1\} \times \{0\}, \{1\} \times \{0,1\} \times \{1\} \times \{0,1\}, \{0,1\} \times \{0\} \times \{1\} \times \{0,1\}, \{0\} \times \{0,1\} \times \{1\} \times \{0,1\}, \{0,1\} \times \{1\} \times \{1\} \times \{0,1\}, \{0\} \times \{0,1\} \times \{1\} \times \{1,2\}, \{0,1\} \times \{0\} \times \{1,2\} \times \{1\}, \{0,1\} \times \{1\} \times \{1,2\} \times \{1\}, \{0,1\} \times \{0,1\} \times \{2\} \times \{1\}, \{2\} \times \{0,1\} \times \{1\} \times \{1,2\}, \{1,2\} \times \{0\} \times \{1,2\} \times \{1\}, \{1,2\} \times \{0,1\} \times \{1\} \times \{1\}, \{1,2\} \times \{0,1\} \times \{2\} \times \{1\}, \{2\} \times \{1,2\} \times \{1\} \times \{1,2\}, \{1\} \times \{1,2\} \times \{1\} \times \{1,2\}, \{1,2\} \times \{2\} \times \{1,2\} \times \{1\}, \{1,2\} \times \{1,2\} \times \{2\} \times \{1\}, \{2\} \times \{1,2\} \times \{0\} \times \{1,1\}, \{1,2\} \times \{1\} \times \{0,1\} \times \{1\}, \{1,2\} \times \{2\} \times \{0,1\} \times \{1\}, \{1,2\} \times \{1,2\} \times \{0\} \times \{1\}, \{0\} \times \{1,2\} \times \{0\} \times \{1,1\}, \{0,1\} \times \{1,2\} \times \{0\} \times \{1\}, \{0,1\} \times \{1,2\} \times \{1\} \times \{1\}, \{0,1\} \times \{2\} \times \{0,1\} \times \{1\}, \{0,1\} \times \{1\} \times \{0\} \times \{0,1\}, \{0\} \times \{1\} \times \{0,1\} \times \{0,1\}, \{1\} \times \{0,1\} \times \{0\} \times \{0,1\}, \{1\} \times \{0,1\} \times \{0\} \times \{1,1\}, \{0\} \times \{0\} \times \{0,1\} \times \{0,1\}, \{0,1\} \times \{0\} \times \{0\} \times \{0,1\}, \{0,1\} \times \{0\} \times \{0,1\} \times \{1\}\}.$

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