Minimal simple sets: A new concept for topology-preserving transformations

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Homotopic skeletonisation in cubic grids

Homotopic skeletonisation: used to transform an object without topology modification.
In discrete grids ($\mathbb{Z}^2$, $\mathbb{Z}^3$, $\mathbb{Z}^4$), defined and implemented thanks to the notion of simple point.

Algorithm

Input: $X \subset \mathbb{Z}^n$
Output: $S \subseteq X$ *(S topologically equivalent to X)*

Let $S = X$
while $\exists x \in S, simple(x, S)$ **do**
  Choose $x \in S, simple(x, S)$ *according to some criterion*
  $S = S \setminus \{x\}$
end while

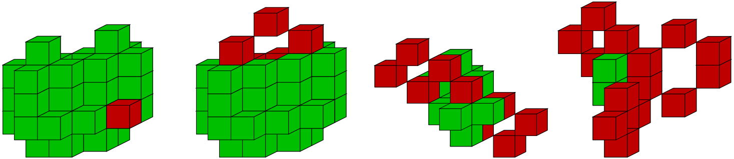
Simple points do not guaranty “correct results”

Proposition

There exist objects $X, Y \subset \mathbb{Z}^n$ such that:

- *X does not contain any simple points;*
- *$Y \subset X$ is however topologically equivalent to X.*

Conclusion: reduction algorithms only based on simple points may fail to lead to “minimal results”.



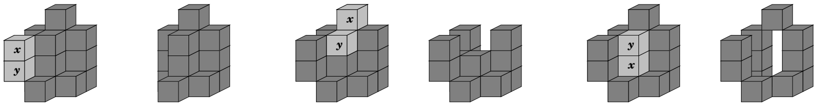
A solution: the simple sets

Definition (Unformal and partial)

Let $X \subset \mathbb{Z}^n$. Let $S \subset X$. If S can be removed from X "without altering its topology", we say that S is a simple set for X.

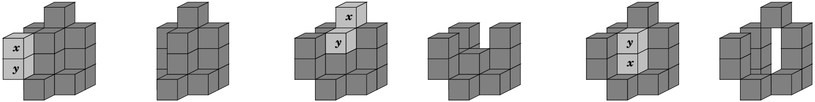
Remark

The notion of simple set extends the notion of simple point (simple points are "singular" simple sets).



Different kinds of simple sets. . .

Three simple sets $S = \{x, y\}$ for the same object X.



- Left: simple set based on P-simple points: x (resp. y) is simple for X; x (resp. y) is simple for $X \setminus \{y\}$ (resp. $X \setminus \{x\}$).
- Middle: simple set based on “successively” simple points: x is simple for X; y is simple for $X \setminus \{x\}$ but not for X.
- Right: simple set without simple points: x (resp. y) is not simple for X but $\{x, y\}$ is simple for X.

Purpose

The last set P is a counter-example to following conjecture.

Conjecture (Kong *et al.*, 1990)

Suppose $X' \subseteq X$ *are finite subsets of* $\mathbb{Z}^3$ *and* X *is collapsible to* X'. *Then there are sets* $X_1, X_2, \ldots, X_n$ *with* $X_1 = X$, $X_n = X'$ *and, for* $0 < i < n$, $X_{i+1} = X_i \setminus \{x_i\}$ *where* x_i *is a simple point of* X_i.

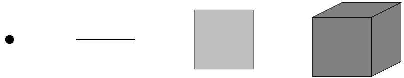
- Simple points are not a sufficient concept to handle simple sets.
- Purpose: study of simple sets (and especially the *minimal* ones, which do not include simple points).
- Study in the context of cubical complexes (more general than $\mathbb{Z}^n$).

Cubical complexes: basic notions

$\mathbb{F}_0^1 = \{\{a\} \mid a \in \mathbb{Z}\}$.
$\mathbb{F}_1^1 = \{\{a, a+1\} \mid a \in \mathbb{Z}\}$.

Definition (Face)

- A *(m-)face* of $\mathbb{Z}^n$ is the Cartesian product of m elements of $\mathbb{F}_1^1$ and $(n-m)$ elements of $\mathbb{F}_0^1$.
- The *dimension* of f is $\dim(f) = m$.
- $\mathbb{F}^n$ is the set composed of all m-faces of $\mathbb{Z}^n$ ($m = 0$ to n).

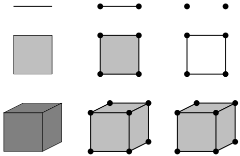


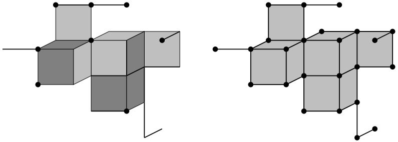
Cubical complexes: basic notions

Definition (Closure)

Let f be a face in $\mathbb{F}^n$.

- $\hat{f} = \{g \in \mathbb{F}^n \mid g \subseteq f\}$ is the set of the *faces* of f.
- $\hat{f}^* = \hat{f} \setminus \{f\}$ is the set of the *proper faces* of f.
- $F^- = \bigcup\{\hat{f} \mid f \in F\}$ is the *closure* of F (F finite).





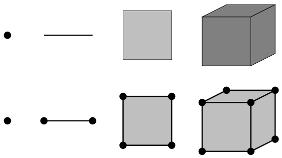
Cubical complexes: basic notions

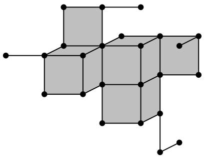
Definition (Cell)

A set $F \subset \mathbb{F}^n$ is a an *(m-)cell* if there exists an m-face $f \in F$, such that $F = \hat{f}$.

Definition (Complex)

A set $F \subset \mathbb{F}^n$ (F finite) is a *complex* if for any $f \in F$, we have $\hat{f} \subseteq F$, *i.e.*, if $F = F^-$. We write $F \preceq \mathbb{F}^n$.





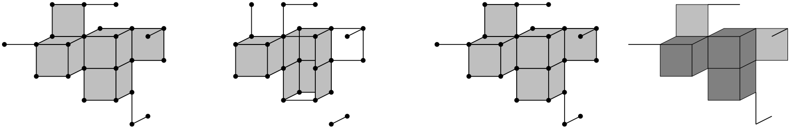
Cubical complexes: basic notions

Definition (Subcomplex)

A subset $G \subseteq F \preceq \mathbb{F}^n$ which is also a complex is a *subcomplex* of F. We write $G \preceq F$.

Definition (Facet)

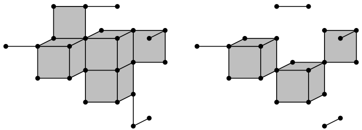
A face $f \in F \preceq \mathbb{F}^n$ is a *facet* of F if there is no $g \in F$ such that $f \in \hat{g}^*$. F^+ is the set of all facets of F.



Cubical complexes: basic notions

Definition (Principal subcomplex)

If $G \preceq F \preceq \mathbb{F}^n$ and $G^+ \subseteq F^+$, then G is a *principal subcomplex* of F. We write $G \sqsubseteq F$ (and $G \sqsubset F$ if $G \neq F$).



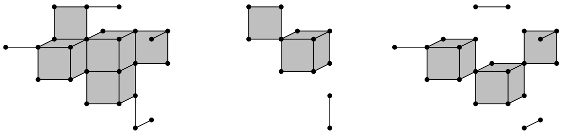
Definition (Dimension, purity)

The *dimension* of $\emptyset \neq F \preceq \mathbb{F}^n$ is $\dim(F) = \max\{\dim(f) \mid f \in F^+\}$. F is a *pure* complex if for all $f \in F^+$, we have $\dim(f) = \dim(F)$.

Cubical complexes: basic notions

Definition (Detachment)

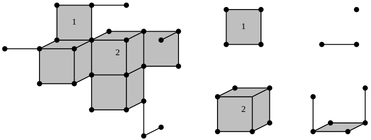
Let $G \preceq F \preceq \mathbb{F}^n$. The complex $F \oslash G = (F^+ \setminus G^+)^-$ is the *detachment of G from F*.



Cubical complexes: basic notions

Definition (Attachment)

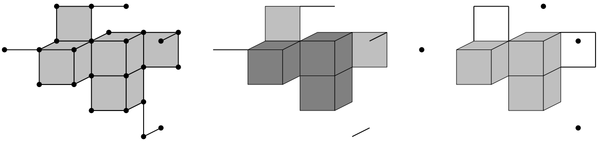
Let $G \preceq F \preceq \mathbb{F}^n$. The complex $Att(G, F) = G \cap (F \oslash G)$ is the *attachment of G to F*.



Cubical complexes: topological notions

Definition (Elementary collapse)

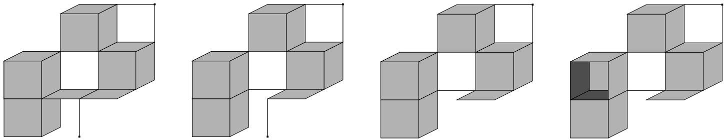
Let $f \in F^+$, with $F \preceq \mathbb{F}^n$. If $g \in \hat{f}^*$ is such that f is the only face of F which strictly includes g, then we say that (f, g) is a *free pair* for F. If (f, g) is a free pair for F, the complex $F \setminus \{f, g\} \preceq F$ is an *elementary collapse* of F.



Cubical complexes: topological notions

Definition (Collapse)

Let $G \preceq F \preceq \mathbb{F}^n$. We say that F *collapses onto* G, and we write $F \searrow G$, if there exists a sequence of complexes $\langle F_i \rangle_{i=0}^{t}$ $(t \geq 0)$ such that $F_0 = F$, $F_t = G$, and F_i is an elementary collapse of F_{i-1}, for all $i \in [1, t]$. The sequence $\langle F_i \rangle_{i=0}^{t}$ is a *collapse sequence from F to G*.



Remark

Collapsing is an homotopy-preserving operation.

Simple set

A set $G \preceq F$ is simple if there is a topology-preserving deformation (*i.e.* a collapse) of F over itself onto the relative complement of G in F.

Definition

Let $G \preceq F \preceq \mathbb{F}^n$. We say that G is *simple for* F if $F \searrow F \oslash G \neq F$. Such a subcomplex G is called a *simple subcomplex of* F or a *simple set* (SS) *for* F.

Minimal simple set

Minimal simple sets (MSSs) are a sub-family of simple sets presenting minimality properties.

Definition

Let $G \preceq F \preceq \mathbb{F}^n$. The complex G is a *minimal simple subcomplex* (or a *minimal simple set*) *for* F if G is a simple set for F and G is minimal (w.r.t. $\preceq$) for this property (*i.e.* $\forall H \preceq G$, H is simple for $F \Rightarrow H = G$).

Remark

(i) The existence of a simple set implies the existence of a minimal simple set.
(ii) A minimal simple set is easier to characterise than a simple set.

Simple cells

Simple cells are simple sets with exactly one facet. The following definition can be seen as a discrete counterpart of the one given by Kong (DGCI'97).

Definition

Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $f \in F^+$ be a facet of F. The cell $\hat{f} \sqsubseteq F$ is a simple cell for F if $F \searrow F \oslash \hat{f}$.

Remark

Simple cells are minimal simple sets.

Simple-equivalence

From the notion of simple cell, we can define the concept of *simple-equivalence*. . .

Definition

Let $F, F' \preceq \mathbb{F}^n$. We say that F and F' are *simple-equivalent* if there exists a sequence of sets $\langle F_i \rangle_{i=0}^{t}$ ($t \geq 0$) such that $F_0 = F$, $F_t = F'$, and for any $i \in [1, t]$, we have either:
(i) $F_i = F_{i-1} \oslash H_i$, where $H_i \sqsubseteq F_{i-1}$ is a simple cell for F_{i-1}; or
(ii) $F_{i-1} = F_i \oslash H_i$, where $H_i \sqsubseteq F_i$ is a simple cell for F_i.

Lumps

...and from the notion of simple-equivalence, we can define the notion of *lump*.

Definition

Let $F' \preceq F \preceq \mathbb{F}^n$ such that F and F' are simple-equivalent. If F does not include any simple cell outside F', then we say that F is a *lump relative to F'*, or simply a *lump*.

Remark

A lump F relative to F', although not including any simple cell which can be detached to provide a monotonic reduction converging onto F', can sometimes (but not necessarily...) include simple sets.

3-D minimal simple pairs in 3-D space

- First result on "non-trivial" 3-D SSs in 3-D spaces.
- Characterisation of *minimal simple pairs* (MSPs) in pure 3-D complexes ($\sim \mathbb{Z}^3$, with a $(26, 6)$-adjacency), in linear time.
- Algorithmic improvements ("simple points + simple pairs" better than "simple points").

Proposition (DGCI'08, JMIV)

The set $P \sqsubseteq F \sqsubseteq \mathbb{F}^3$ is a minimal simple pair for F if and only if all the following conditions hold:
(i) the intersection of the two facets of P is a 2-face,
(ii) $\forall g \in P^+, |C[Att(\hat{g}, F)]| = 1$,
(iii) $\forall g \in P^+, \chi(Att(\hat{g}, F)) \leq 0$,
(iv) $|C[Att(P, F)]| = 1$,
(v) $\chi(Att(P, F)) = 1$.

Decomposition of SSs into simple cells

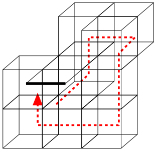
Proposition

Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubseteq F$ be a simple set for F. If $n \leq 2$ or $\dim(G) \leq 1$, then $\exists H \sqsubseteq G$ such that H is a simple cell for F.

"Simple sets can be handled by considering simple cell for $n \leq 2$ or $\dim(G) \leq 1$ (*i.e.* any SS is necessarily composed of simple cells)".

Remark

This is no longer be true for $n \geq 3$ and $\dim(G) \geq 2$.



Decomposition of SSs into MSSs

Proposition

Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubseteq F$ be a (non-minimal) simple set for F such that $\dim(G) \leq 2$. Then $\forall H \sqsubset G$ such that H is a minimal simple set for F, $G \oslash H$ is a simple set for $F \oslash H$.

"Simple sets of dimension lower than 2 can be broken by successive removal of *any* sequence of minimal simple sets, independently of the dimension of the space where they lie".

Remark

This is no longer be true for $\dim(G) \geq 3$ (cf. Bing's house).

Contribution

Done:

- Sound definitions of SSs, MSSs, lumps, etc. (tools for homotopy preserving transforms);
- a characterisation of 3-D MSPs in pure 3-D objects.

Work in progress (published soon, hopefully):

- Some general results on SSs/MSSs (in n-D spaces);
- A complete study of 2-D SSs/MSSs in "general" 2-D spaces (*i.e.* 2-D pseudomanifolds);
- A complete study of 2-D SSs/MSSs in n-D spaces (with a characterisation of MSSs in linear time!);
- A characterisation of 3-D MSPs in n-D spaces.

Further works

Coming latter (or not. . .):

- A complete study of 3-D SSs/MSSs in 3-D spaces?
- 4-D and more?
- A study of SSs with alternative definitions (cf. Bing's house, etc.)?
- *Ad lib.*

Related publications

- N. Passat, M. Couprie, G. Bertrand.
  Topological monsters in $\mathbb{Z}^3$: A non-exhaustive bestiary.
  ISMM 2007, Vol. 2, pp. 11-12.
- N. Passat, M. Couprie, G. Bertrand.
  Minimal simple pairs in the cubic grid.
  DGCI 2008, LNCS, Vol. 4992, pp. 165-176.
- N. Passat, M. Couprie, G. Bertrand.
  Minimal simple pairs in the 3-D cubic grid.
  Journal of Mathematical Imaging and Vision. In Press.

Related publications

Available soon (as research reports in the next few weeks/months)

- N. Passat, M. Couprie, L. Mazo, G. Bertrand.
  An introduction to simple sets.
  Research Report IGM2008-xx.
- N. Passat, M. Couprie, L. Mazo, G. Bertrand.
  Simple sets in 2-D pseudomanifolds.
  Research Report IGM2008-xx.
- L. Mazo, N. Passat, M. Couprie, G. Bertrand.
  2-D simple sets in n-D cubic grids.
  Research Report IGM2008-xx.
- N. Passat, M. Couprie, G. Bertrand.
  3-D minimal simple pairs in n-D cubic grids.
  Research Report IGM2008-xx.

Thank you for your attention.

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