Minimal simple sets: A new concept for topology-preserving transformations

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Abstract Preserving topological properties of binary objects during thinning procedures is an important issue in the field of image analysis. In this context, we present the new notion of *simple set* which extends the well-known notion of simple point. Similarly to simple points, simple sets have the property that the homotopy type of the object in which they lie is not changed when such sets are removed. This paper defines and justifies the concept of simple sets and of a sub-family of such sets called *minimal simple sets*. It also briefly describes some first results and work in progress on this subject.

Keywords Topology preservation, binary image reduction, n-D cubical complexes.

1 Motivation

Topology-preserving operators are used to transform an object while leaving unchanged its topological characteristics. In discrete grids ($\mathbb{Z}^2$ or $\mathbb{Z}^3$), they can be defined and efficiently implemented thanks to the notion of simple point [7, 1]. We say that a discrete object Y obtained from X by iterative removal of simple points until stability is a *homotopic skeleton* of X. Such a set fulfils a property of minimality: Y is minimal in the sense that it no longer contains simple points. However, we could formulate a stronger minimality requirement: informally, Y should not strictly include any set Z "topologically equivalent" to X.

In $\mathbb{Z}^2$ a full rectangle X is topologically equivalent to a single point, thus all homotopic skeletons of X should be singletons. Rosenfeld proved in [17] that any homotopic skeleton of X is indeed reduced to a single point. However, in dimensions $n \geq 3$, this property does not hold: if X is a full cube, we may find a homotopic skeleton of X which is not reduced to a single point. A classical example of such a skeleton is Bing's house with two rooms [3] (see Figure 1).

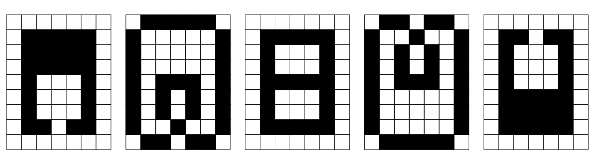
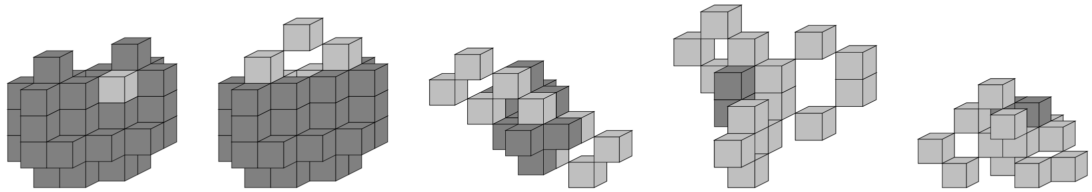


Figure 1: A Bing's house (considered in 26-adjacency), decomposed into slices for visualisation.

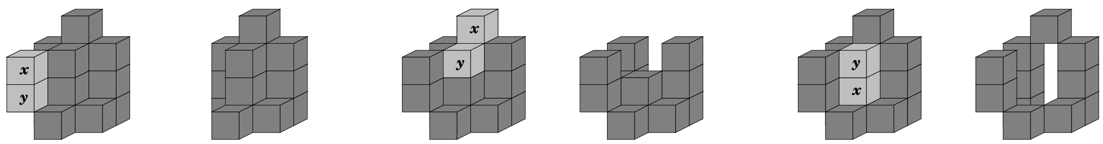
It could be argued that objects like Bing's houses are unlikely to appear while processing "real" images, because of their complex shape and their size. However, there exists a large class of objects - of any possible topology - presenting similar properties [13], some of them being quite small (see Figure 2): such objects (precisely defined in Definition 10) will be called *lumps*.

Two questions now arise: is it possible to detect when a thinning procedure gets stuck on a lump, and then, is it possible to find a way towards a homotopic skeleton? For performing the latter task, a solution consists of identifying subsets of X which can be removed without altering its topology: such subsets will be called *simple sets*.



(a) (b) (c) (d) (e)

Figure 2: Examples of lumps of various topologies (considered in 26-adjacency). Some simple-equivalent subsets (see Sec. 4) are depicted in light grey.



(a) (b) (c)

Figure 3: A 3-D set X (in 26-adjacency) including several kinds of simple sets S_i (in light grey). (a) Left: $S_1 \subset X$, right: $X \setminus S_1$. (b) Left: $S_2 \subset X$, right: $X \setminus S_2$. (c) Left: $S_3 \subset X$, right: $X \setminus S_3$.

Intuitively, a simple set can be defined as a subset $S \subset X$ whose removal from X "does not alter the topology of X". Following this informal definition, the sets S_i of Figure 3 are simple for the set X in which they lie.

The set S_1 is composed of two points x and y both simple for X. The removal of x (resp. y) from X then does not alter its topology. Moreover, this is also true for the iterative (or parallel) removal of x and y from X. Such points, called P-simple points, have been characterised in [2]. Any set S composed of points being P-simple for a set X can be removed from X without altering its topology: such a set is then a simple set for X.

The set S_2 is composed of two points x and y such that x is simple for X while y is simple for $X \setminus \{x\}$ but not for X. Consequently, the iterative removal of x, then y from X does not alter its topology. The sets composed - as S_2 - of "successively" simple points have been described and/or studied in [16, 10, 5, 4]. Any set S composed of such successively simple points for a set X is a simple set for X.

The set S_3 is composed of two points x and y which are both non-simple for X. However, S_3 can be removed from X in a way which corresponds to a reduction of X consisting in iteratively removing "successive parts of x and y" until obtaining $X \setminus S_3$ without altering its topology. Such a set S_3 is then a simple set for X, despite the fact that it does not contain any simple point. The set S_3 (which can be found in Figure 2(a) and (b)) provides a counter-example to the conjecture proposed by Kong *et al.* in [8], emphasising the fact that simple points are not sufficient to completely deal with the problem of topology-preserving reduction of discrete images.

Conjecture 1 *([8], Conjecture 1, p. 383) Suppose $X' \subseteq X$ are finite subsets of $\mathbb{Z}^3$ and X is collapsible to X'. Then there are sets $X_1, X_2, \ldots, X_n$ with $X_1 = X$, $X_n = X'$ and, for $0 < i < n$, $X_{i+1} = X_i \setminus \{x_i\}$ where x_i is a simple point of X_i.*

Any subset $S \subset X$ which may be removed from X by successive "partial removal" of its points (corresponding to the notion of *collapse* described in Sec. 2) is then simple. The notion of simple set considered here[1] will be defined in this way, which extends the simple sets defined in [16, 10, 5, 4] or from [2], since they may now be composed of simple points but also of non-simple points.

The purpose of this paper is to define the notion of minimal simple sets in a sound fashion (Secs. 3, 4), and to describe first results and work in progress (Sec. 5). Definitions and results will be

[1] Another definition of simple sets (authorising a simple set to be removed by a non-monotonic transform) could be proposed; see [14] for a discussion about the algorithmic interest of both definitions.

proposed in the framework of cubical complexes [9] (Sec. 2) which enables to model $\mathbb{Z}^n$ ($n \in \mathbb{N}^*$), retrieving the main notions and results of digital topology (such as the notion of simple point), but also to define more general objects composed of parts of various dimensions and structured on regular grids.

2 Cubical complexes

Let $\mathbb{Z}$ be the set of integers. We consider the families of sets $\mathbb{F}_0^1$, $\mathbb{F}_1^1$, such that $\mathbb{F}_0^1 = \{\{a\} \mid a \in \mathbb{Z}\}$, $\mathbb{F}_1^1 = \{\{a, a+1\} \mid a \in \mathbb{Z}\}$. A subset f of $\mathbb{Z}^n$ ($n \geq 1$) which is the Cartesian product of exactly m elements of $\mathbb{F}_1^1$ and $(n-m)$ elements of $\mathbb{F}_0^1$ is called a *face* or an *m-face* of $\mathbb{Z}^n$, m is the *dimension* of f, and we write $\dim(f) = m$.

We denote by $\mathbb{F}^n$ the set composed of all m-faces of $\mathbb{Z}^n$ ($m = 0$ to n). Let f be a face in $\mathbb{F}^n$. We set $\hat{f} = \{g \in \mathbb{F}^n \mid g \subseteq f\}$, and $\hat{f}^* = \hat{f} \setminus \{f\}$. Any $g \in \hat{f}$ is a *face* of f, and any $g \in \hat{f}^*$ is a *proper face* of f. If F is a finite set of faces of $\mathbb{F}^n$, we write $F^- = \bigcup\{\hat{f} \mid f \in F\}$, F^- is the *closure* of F.

A set F of faces of $\mathbb{F}^n$ is a *cell* or an *m-cell* if there exists an m-face $f \in F$, such that $F = \hat{f}$. The *boundary* of a cell $\hat{f}$ is the set $\hat{f}^*$. A finite set F of faces of $\mathbb{F}^n$ is a *complex* (in $\mathbb{F}^n$) if for any $f \in F$, we have $\hat{f} \subseteq F$, *i.e.*, if $F = F^-$. Any subset G of a complex F which is also a complex is a *subcomplex* of F. If G is a subcomplex of F, we write $G \preceq F$. If F is a complex in $\mathbb{F}^n$, we also write $F \preceq \mathbb{F}^n$.

A face $f \in F$ is a *facet* of F if there is no $g \in F$ such that $f \in \hat{g}^*$. We denote by F^+ the set composed of all facets of F. Observe that $(F^+)^- = F^-$ and thus, that $(F^+)^- = F$ whenever F is a complex.

If G is a subcomplex of F, and $G^+ \subseteq F^+$, then G is a *principal subcomplex* of F, and we write $G \sqsubseteq F$. To explicitly express the fact that $G \sqsubseteq F$ and $G \neq F$, we write $G \sqsubset F$.

The *dimension* of a non-empty complex F in $\mathbb{F}^n$ is defined by $\dim(F) = \max\{\dim(f) \mid f \in F^+\}$. We say that F is an *m-complex* if $\dim(F) = m$. We say that F is a *pure* complex if for all $f \in F^+$, we have $\dim(f) = \dim(F)$.

Definition 2 (Detachment) *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \preceq F$ be a subcomplex of F. We set $F \oslash G = (F^+ \setminus G^+)^-$. The set $F \oslash G$ is a complex which is the* detachment *of G from F.*

Definition 3 (Attachment) *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \preceq F$ be a subcomplex of F. The* attachment *of G to F is the complex defined by $Att(G, F) = G \cap (F \oslash G)$.*

Two distinct faces f and g of $\mathbb{F}^n$ are *adjacent* if $f \cap g \neq \emptyset$. Let $F \preceq \mathbb{F}^n$ be a non-empty complex. A sequence $(f_i)_{i=0}^s$ ($s \geq 0$) of faces of F is a *path in F (from f_0 to f_s)* if f_i and f_{i+1} are adjacent, for all $i \in [0, s-1]$. We say that F *is connected* if, for any two faces $f, g \in F$, there is a path from f to g in F. We say that G is a *connected component of* F if $G \sqsubseteq F$, G is connected and if G is maximal for these two properties (*i.e.*, we have $H = G$ whenever $G \sqsubseteq H \sqsubseteq F$ and H is connected). We denote by $C[F]$ the set of all the connected components of F. We set $C[\emptyset] = \emptyset$.

We now define the notion of *collapsing*, which is a well-known operation of topology that preserves homotopy type. The notion of simple set will be defined thanks to this operation.

Definition 4 (Elementary collapse) *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $f \in F^+$. If $g \in \hat{f}^*$ is such that f is the only face of F which strictly includes g, then we say that (f, g) is a* free pair *for F. If (f, g) is a free pair for F, the complex $F \setminus \{f, g\} \preceq F$ is an* elementary collapse *of F.*

Definition 5 (Collapse) *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \preceq F$ be a subcomplex of F. We say that F* collapses onto G*, and we write $F \searrow G$, if there exists a sequence of complexes $\langle F_i \rangle_{i=0}^t$ ($t \geq 0$) such that $F_0 = F$, $F_t = G$, and F_i is an elementary collapse of F_{i-1}, for all $i \in [1, t]$. The sequence $\langle F_i \rangle_{i=0}^t$ is a* collapse sequence from F to G.

3 Minimal simple sets

Intuitively a set $G \preceq F$ is simple if there is a topology-preserving deformation of F over itself onto the relative complement of G in F.

Definition 6 *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \preceq F$ be a subcomplex of F. We say that G is* simple for F *if $F \searrow F \oslash G \neq F$. Such a subcomplex G is called a* simple subcomplex of F *or a* simple set for F.

Minimal simple sets constitute a sub-family of simple sets presenting minimality properties.

Definition 7 *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \preceq F$ be a subcomplex of F. The complex G is a* minimal simple subcomplex *(or a* minimal simple set*) for F if G is a simple set for F and G is minimal (w.r.t. $\preceq$) for this property (*i.e. *$\forall H \preceq G$, H is simple for $F \Rightarrow H = G$).*

The notion of minimal simple set is useful since (i) the existence of a simple set necessarily implies the existence of at least one minimal simple set, and (ii) by definition, a minimal simple set is necessarily easier (or, at least, not harder) to characterise than a "general" simple set. In particular, we can hope that in several cases (depending on the value of n, and/or on the dimension of F, for example), the study of minimal simple sets could be sufficient to deal with the problem of detaching *all* simple sets from a complex.

Simple cells are (minimal) simple sets containing exactly one facet. The following definition of simple cells can be seen as a discrete counterpart of the one given by Kong in [6].

Definition 8 *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $f \in F^+$ be a facet of F. The cell $\hat{f} \sqsubseteq F$ is a simple cell for F if $F \searrow F \oslash \hat{f}$.*

4 Simple-equivalence and lumps

From the notion of simple cell, we can define the concept of *simple-equivalence*, leading to the notion of *lump*.

Definition 9 *Let $n \geq 1$. Let $F, F' \preceq \mathbb{F}^n$ be cubical complexes. We say that F and F' are* simple-equivalent *if there exists a sequence of sets $\langle F_i \rangle_{i=0}^t$ ($t \geq 0$) such that $F_0 = F$, $F_t = F'$, and for any $i \in [1, t]$, we have either:*
(i) $F_i = F_{i-1} \oslash H_i$, where $H_i \sqsubseteq F_{i-1}$ is a simple cell for F_{i-1}; or
(ii) $F_{i-1} = F_i \oslash H_i$, where $H_i \sqsubseteq F_i$ is a simple cell for F_i.

Definition 10 *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $F' \preceq F$ be a subcomplex of F such that F and F' are simple-equivalent. If F does not include any simple cell outside F', then we say that F is a* lump relative to F', *or simply a* lump.

As stated in Sec. 1, a lump F relative to F', although not including any simple cell which can be detached to provide a monotonic reduction converging onto F', can sometimes (but not necessarily) include simple sets.

5 Results and work in progress

The study of simple sets presents an interest from a theoretical point of view but also from a practical one, since a knowledge - and easy characterisations - of such sets can lead to the development of efficient reduction algorithms enabling, in particular, to improve the results provided by algorithms only based on simple points.

The first obtained results deal with the smallest non-trivial 3-D simple sets in 3-D spaces (corresponding to the "classical" case of binary images in $\mathbb{Z}^3$ considered with a 26-adjacency for objects), called the *simple pairs*, and in particular the minimal ones (*i.e.* those composed only of non-simple points). The set $P = \{x, y\}$ of Figure 3(c) is an example of minimal simple pair. The results presented in [12] propose a characterisation (see Proposition 11) of minimal simple pairs which enables to detect all of them in a set X with a linear algorithmic complexity $O(|X|)$.

Proposition 11 *The set $P \sqsubseteq F \sqsubseteq \mathbb{F}^3$ is a minimal simple pair for F if and only if all the following conditions hold (note that χ is the Euler characteristic):*

$$\begin{aligned} &\textit{the intersection of the two facets of } P \textit{ is a } 2\textit{-face},\\ &\forall g \in P^+, |C[Att(\hat{g}, F)]| = 1,\\ &\forall g \in P^+, \chi(Att(\hat{g}, F)) \leq 0,\\ &|C[Att(P, F)]| = 1,\\ &\chi(Att(P, F)) = 1. \end{aligned}$$

Experiments have shown that reduction algorithms based on both simple points and simple pairs provide significantly better results (*i.e.* with less lumps) than algorithms based on simple points only (see [12], Appendix A).

These first works on simple sets have been related to 3-D images since the applications which led to consider this new notion were actually devoted to such 3-D (medical) data. Since then, other works on simple sets in 2-D spaces [15] ("classical" or embedded in spaces of higher dimensions) and of 2-D simple sets in n-D ($n \geq 2$) spaces [11] have started. Their results will be presented soon, and will enable (i) to generalise known results in 2-D and (ii) to characterise new families of objects (such as the minimal simple set depicted in Figure 4) presenting quite interesting properties.

Parallely, a study of properties of simple sets and minimal simple sets which are general (*i.e.* not sufficient to lead to a characterisation) but true in any dimension, is currently being considered, and will be presented in [14]. In this study, it will notably be proved that minimal simple sets verify several structural properties, including *principalness* and *connectedness*, and forbid certain configurations, especially related to their attachment. By fusion of results proved in [14, 15, 11] we will also obtain the following propositions.

Proposition 12 *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubseteq F$ be a simple set for F. If $n \leq 2$ or $\dim(G) \leq 1$, then $\exists H \sqsubset G$ such that H is a simple cell for F.*

Proposition 13 *Let $n \geq 1$. Let $F \preceq \mathbb{F}^n$ be a cubical complex. Let $G \sqsubseteq F$ be a simple set for F such that $\dim(G) \leq 2$. Then $\forall H \sqsubset G$ such that H is a minimal simple set for F, $G \oslash H$ is a simple set for $F \oslash H$.*

Proposition 12 states that the notion of simple set can be fully handled thanks to the notion of simple cell/point for $n \leq 2$ or $\dim(G) \leq 1$ (*i.e.* any simple set is necessarily "composed of" simple cells), which will no longer be true for $n \geq 3$ and $\dim(G) \geq 2$, and will then require to carefully study the notion of simple sets in such conditions. Proposition 13 states that simple

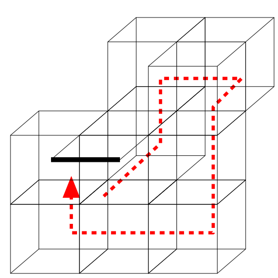


Figure 4: A (pure) 2-D minimal simple set S. The dark line represents the attachment of S to its superset (not depicted here). The arrow enables to visualise the "missing" 2-faces of the object.

sets of dimension lower than 2 can be "broken" by successive removal of *any* sequence of minimal simple sets, independently of the dimension of the space where they lie, which is a quite interesting result, that will - hopefully - enable to develop efficient reduction techniques for such sets.

Our last efforts have focused on the study of (minimal) simple sets of "low dimension". Once these studies will be complete, we will consider at new the study of 3-D (and hopefully higher dimension) simple sets, which will probably be a quite hard - and quite interesting - issue.

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