An extension of component-trees to partial orders

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Introduction

The well-known structure of *component-tree* models some structural characteristics of an image by considering its "binary slices" obtained by threshold operations at successive levels. We explore new ways for the use of component-trees by not only considering images taking their values in completely-ordered sets ("grey-level" images), but more generally in *partially-ordered* ones, such as colour images.

Definitions, notations

Let E be a discrete set. Let $(V, \leq)$ be a partially ordered set, let $\perp$ be its infimum. An image on E taking its values in V is noted $I : E \to V$ (or $I \in V^E$). The set of the connected components of a binary set $X \subseteq E$ is noted $\mathcal{C}[X]$. The threshold operator at the value $v \in V$ is noted $X_v : V^E \to \mathcal{P}(E)$. We define the cylinder function $C_{X,v} : E \to V$ by $C_{X,v}(x) = v$ if $x \in X$ and $\perp$ otherwise. An image $I \in V^E$ can be expressed as $I = \bigvee_{v \in V} \bigvee_{X \in \mathcal{C}[X_v(I)]} C_{X,v}$.
Let $\mathcal{K} = \bigcup_{v \in V}(\mathcal{C}[X_v(I)] \times \{v\})$. Let $\subseteq_{\mathcal{K}}$ be the relation defined by $(X, v_X) \subseteq_{\mathcal{K}} (Y, v_Y)$ if $X \subset Y$ or $(X = Y$ and $v_Y \leq v_X)$. The inclusion relation $\subseteq_{\mathcal{K}}$ is a partial order on $\mathcal{K}$. (By abuse of notation, we will also note $X \in \mathcal{K}$ for $(X, v_X) \in \mathcal{K}$, and $\subseteq$ for $\subseteq_{\mathcal{K}}$.)
Let $B = X_v(I) \subseteq E$. Let $C \in \mathcal{C}[B]$ be a connected component of B. There may exist *several* values $v_i \leq v$ $(i \in \mathbb{N})$, such that $C \subseteq C_i \in \mathcal{C}[X_{v_i}(I)]$, and v_i is maximal w.r.t. $\leq$ for these properties. Consequently, the Hasse diagram of the partially-ordered set $(\mathcal{K}, \subseteq)$ *is not necessarily a tree*. Such a diagram (called *component-graph* in the sequel) has however a supremal element (which can be assimilated to its "root"), namely $X_\perp(I) = E$.

Component-graphs

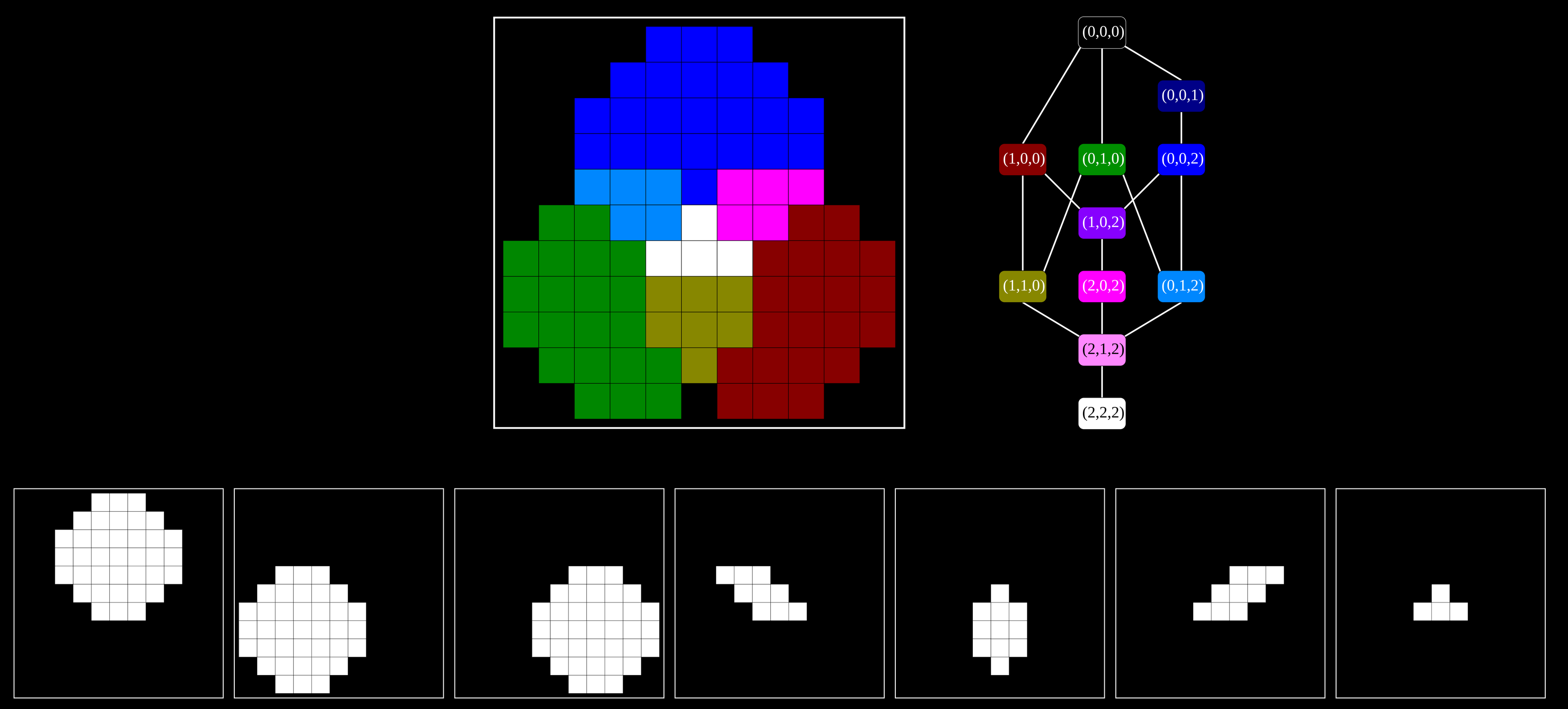
Let $I \in V^E$. The *component-graph* of I is the "rooted" oriented graph $(\mathcal{K}, L, R)$ defined such that $(\mathcal{K}, L)$ is the Hasse diagram of the partially-ordered set $(\mathcal{K}, \subseteq)$, and $R = \sup(\mathcal{K}, \subseteq) = X_\perp(I) = E$. $\mathcal{K}$, R and L are the set of the *nodes*, the *root* and the set of the *edges* of the graph, respectively.
Similarly to component-trees, attributes can be stored at each node of a component-graph. The preservation/removal of nodes of the graph w.r.t. such attributes can then lead to filtering strategies. However, the less restrictive properties of component-graphs (by opposition to those of component-trees) do not enable to straightforwardly apply classical filtering strategies designed for component-trees.

Filtering strategy

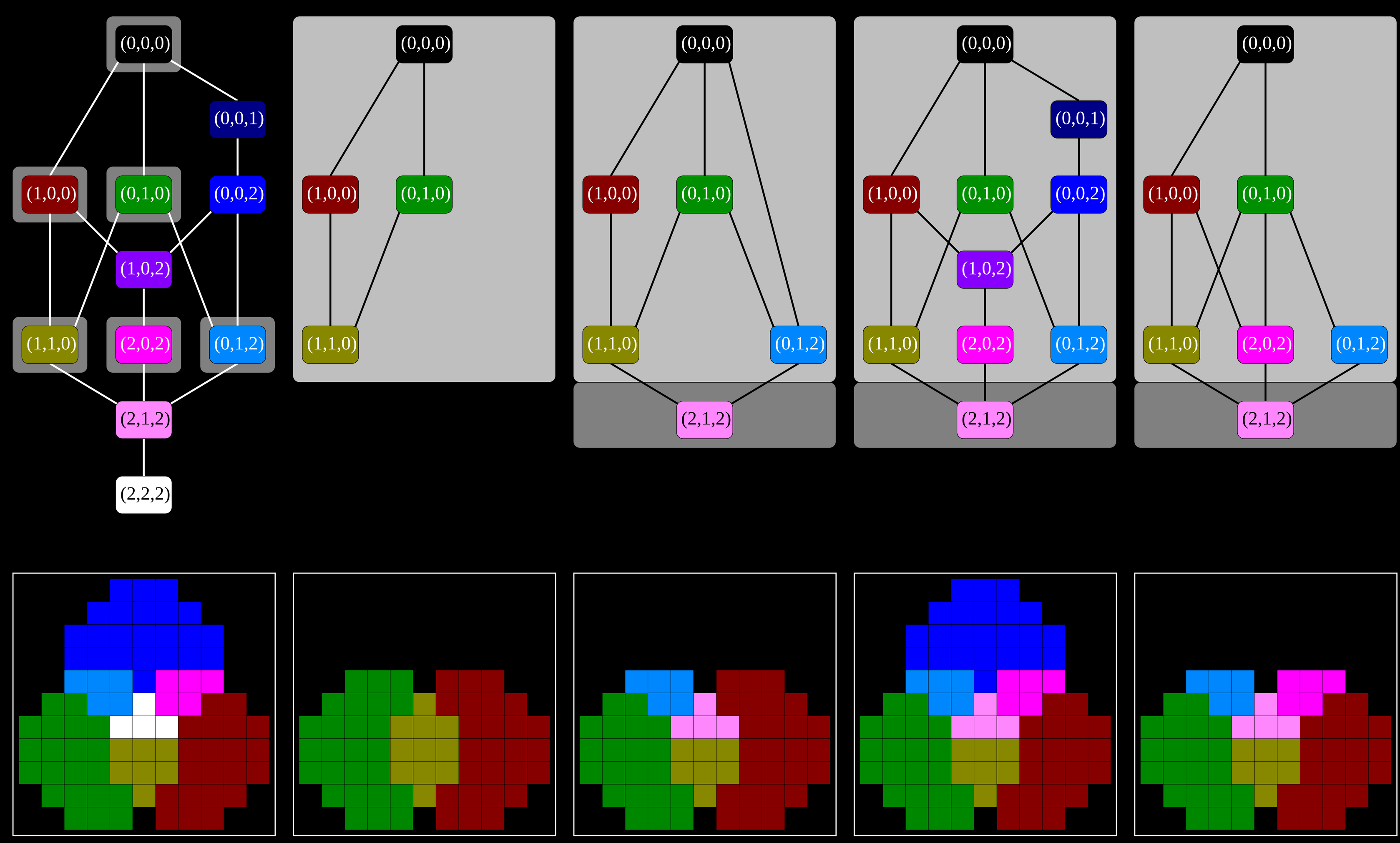
Let ρ be a predicate on $\mathcal{K}$. The following strategy is devoted to the filtering of I by "pruning" its component-graph w.r.t. ρ.
Let $\mathcal{K}' = \mathcal{K} \setminus \{R\}$. In case of *non-increasing* criteria, the following policies can be considered to select a subset $\mathcal{K}_\rho \subseteq \mathcal{K}$ of nodes to be preserved in the component-graph:

$$\begin{aligned}
Min_1: &\quad \mathcal{K}_\rho = \{R\} \cup \{N \in \mathcal{K}' \mid \rho(N) \wedge \forall (P,N) \in L, P \in \mathcal{K}_\rho\}\\
Min_2: &\quad \mathcal{K}_\rho = \{R\} \cup \{N \in \mathcal{K}' \mid \rho(N) \wedge \exists (P,N) \in L, P \in \mathcal{K}_\rho\}\\
Max: &\quad \mathcal{K}_\rho = \{R\} \cup \{N \in \mathcal{K}' \mid \rho(N) \vee \exists (N,C) \in L, C \in \mathcal{K}_\rho\}\\
Direct: &\quad \mathcal{K}_\rho = \{N \in \mathcal{K} \mid \rho(N)\}
\end{aligned}$$

Once $\mathcal{K}_\rho$ has been computed, a new set of edges L_ρ can straightforwardly be defined by building the Hasse diagram of $(\mathcal{K}_\rho, \subseteq)$.
By considering the Min_1 and Min_2 (resp. *Max*, *Direct*) policy(ies) proposed above, in the case of a completely-ordered set, we obviously retrieve the "classical" *Min* (resp. *Max*, *Direct*) policy on component-trees.
However, in the considered case of a partially-ordered set $(V, \leq)$, it may be impossible to reconstruct an image $I_\rho = \bigvee_{(X,v_X) \in \mathcal{K}_\rho} C_{X,v_X}$ defined by $I_\rho(x) = \max_{(X,v_X) \in \mathcal{K}_\rho}\{C_{X,v_X}(x)\}$ for all $x \in E$, since such a maximum may be undefined, due to the partiality of $(V, \leq)$.
In order to deal with this issue, a *coherence recovery* procedure has to be proposed to define unambiguously $\max_{(X,v_X) \in \mathcal{K}_\rho}\{C_{X,v_X}\}$ for all $x \in E$. This procedure consists in adding to the current graph $(\mathcal{K}_\rho, L_\rho)$ a set of nodes and their associated edges, such that the resulting corrected graph $(\mathcal{K}_\rho^c, L_\rho^c)$ enables the generation of a *well-defined* image. Practically, this can be done by computing iteratively - and until stability - $\mathcal{K}_\rho^c$ (initialised to $\mathcal{K}_\rho$) as follows. Choose a point $x \in E$ and two nodes $N_i = (X_i, v_{X_i}) \in \mathcal{K}_\rho^c$ $(i \in \{1,2\})$ (with $N_1 \not\subseteq N_2$, $N_2 \not\subseteq N_1$) such that $x \in X_1 \cap X_2$ and $\forall N = (X, v_X) \in \mathcal{K}_\rho^c \setminus \{N_i\}, x \in X \Rightarrow (N \not\subseteq N_i)$. Then, there exists $N = (X, v_X) \in \mathcal{K} \setminus \mathcal{K}_\rho^c$ such that $x \in X \subseteq X_1 \cap X_2$ and $v_{X_1}, v_{X_2} \leq v_X$. Choose such a node N and set $\mathcal{K}_\rho^c = \mathcal{K}_\rho^c \cup \{N\}$. Once $\mathcal{K}_\rho^c$ has been computed, L_ρ^c is obtained by computing the Hasse diagram of $(\mathcal{K}_\rho^c, \subseteq)$.



Component-graph – First row: (left) a label image $I : [0,10]^2 \to V$ (where $V \subset [0,2]^3$ is composed of 11 distinct values) visualised as an RGB image; (right) the component-graph of I. Second row: some threshold images $X_v(I)$ (from left to right, $v = (0,0,2)$, $(0,1,0)$, $(1,0,0)$, $(0,1,2)$, $(1,1,0)$, $(2,0,2)$, $(2,2,2)$).



Filtering strategies – First column: I and its component-graph, with nodes verifying a given predicate ρ (grey-background). From second to fifth column: light-grey background: graph $(\mathcal{K}_\rho, L_\rho)$ obtained with the Min_1, Min_2, *Max*, and *Direct* policies; light/dark-grey background: corrected graph $(\mathcal{K}_\rho^c, L_\rho^c)$.

Experiments – Left column: original image (Lenna, ©Playboy). First row (centre and right): noisy images (10% and 15% random noise corruption, respectively). Second row (centre and right): filtered images obtained by removing the connected components of size smaller than 10 pixels.