Semi-connections and hierarchies

Olena Tankyevych, Hugues Talbot, Nicolas Passat

Semi-connections and hierarchies*

Olena Tankyevych[1], Hugues Talbot[2], and Nicolas Passat[3]

[1] Université Paris-Est, LISSI, Créteil, France
[2] Université Paris-Est, LIGM, ESIEE-Paris, France
[3] Université de Reims, CReSTIC, France

Abstract. Connectivity is the basis of several methodological concepts in mathematical morphology. In graph-based approaches, the notion of connectivity can be derived from the notion of adjacency. In this preliminary work, we investigate the effects of relaxing the symmetry property from adjacency. In particular, we observe the consequences on the induced connected components, that are no longer organised as partitions but as covers, and on the hierarchies that are obtained from such components. These hierarchies can extend data structures such as component-trees and partition-trees, and the associated filtering and segmentation paradigms, leading to improved image processing tools.

Keywords: connectivity, cover hierarchy, connected operators, filtering.

1 Introduction

Connectivity plays a crucial role in the definition of mathematical morphology. Intuitively, the notion of connectivity serves to decide whether a set is either in one piece, or is split into several ones. This notion has been widely studied [2], from axiomatic definitions [21] to variants such as constrained connectivity [24], second-generation connectivity [15, 22, 7, 11], and hyperconnections [14].

Practically, connectivity in discrete image processing is often handled in graph-based frameworks, via the notion of adjacency [16]. In this context, connectivity has led to the development of data structures based on the partition of discrete spaces, and further on partition hierarchies. Such hierarchies can rely on the image value space, *e.g.*, in the case of component-trees [19, 8], partition-trees [18, 24], or hierarchical watershed [4, 9, 10]. They can also derive from connectivity hierarchies, leading to partition-trees based, *e.g.*, on fuzzy connectedness [17, 1] or second-generation connectivity.

From an applicative point of view, all these concepts have been involved in the development of connected operators [20, 23], devoted in particular to image processing tasks such as filtering or segmentation. In this article, we present a preliminary study on the effects of relaxing the symmetry hypothesis, actually required to define adjacency relations (Secs. 2–3). We observe that partitions then become covers, which leads us to define cover hierarchies instead of partition hierarchies. We prove however that such hierarchies can still be handled as (enriched) tree structures (Sec. 4). This framework generalises standard notions such as component-trees or partition-trees (Sec. 5), and provides solutions for performing more accurate antiextensive filtering tasks (Sec. 6).

* The research leading to these results has received funding from the French *Agence Nationale de la Recherche* (Grant Agreement ANR-2010-BLAN-0205).

2 Background notions: From adjacency to partition hierarchies

We first recall basic definitions and properties related to the concept of adjacency and some induced notions, namely connectedness, partitions and partition hierarchies.

2.1 Adjacency and connectedness

Let Ω be a nonempty finite set. Let $\rightleftharpoons$ be an adjacency (*i.e.*, irreflexive, symmetric, binary) relation on Ω. (We recall that $\rightleftharpoons$ is a subset of $\Omega \times \Omega$.) If $x, y \in \Omega$ satisfy $x \rightleftharpoons y$ (and thus $y \rightleftharpoons x$), we say that x and y are adjacent.

Let $X \subseteq \Omega$ be a nonempty subset of Ω. Let $\rightleftarrows_X$ be the equivalence relation on X induced by the reflexive-transitive closure of the restriction $\rightleftharpoons_X$ of $\rightleftharpoons$ to X. If $x, y \in X$ satisfy $x \rightleftarrows_X y$, we say that x and y are connected (in X). In particular, the equivalence classes of X associated to the relation $\rightleftarrows_X$ are called the connected components of X, and the set of these connected components is noted $\mathcal{C}_{\rightleftarrows}[X]$.

Remark 1 *These definitions are directly linked to classical notions on graphs, considered for the topological modelling of digital images, as introduced,* e.g., *in [16]. In particular,* $(\Omega, \rightleftharpoons)$ *and* $(X, \rightleftharpoons_X)$ *are irreflexive (non-directed) graphs.*

2.2 Partition hierarchies

The notion of connected component, associated to (any subsets of) $(X, \rightleftharpoons_X)$ is important in image analysis. Indeed, the partition $\mathcal{C}_{\rightleftarrows}[X]$ associated to images defined on X, can be considered for filtering and segmentation purpose, by considering approaches that rely on partition hierarchies in the framework of connected operators [20, 23].

In this context, there exist two ways to refine $(X, \rightleftharpoons_X)$, to build partition hierarchies. The first way is to work on X, and to define subsets $Y \subseteq X$, *i.e.*, to progressively constrain the spatial part of $(X, \rightleftharpoons_X)$. Practically, defining $(Y, \rightleftharpoons_Y)$ such that $Y \subseteq X$, implies that $\rightleftharpoons_Y = (\rightleftharpoons_X \cap (Y \times Y))$. The second way is to work on $\rightleftharpoons_X$, and to define (symmetric) subrelations $\overset{\blacktriangle}{\rightleftharpoons}_X \subseteq \rightleftharpoons_X$, inducing connectedness relations $\overset{\blacktriangle}{\rightleftarrows}_X \subseteq \rightleftarrows_X$, *i.e.*, to progressively constrain the structural part of $(X, \rightleftharpoons_X)$.

Remark 2 *In the framework of graphs,* $(Y, \rightleftharpoons_Y)$ *is a subgraph of* $(X, \rightleftharpoons_X)$*, and* $(X, \overset{\blacktriangle}{\rightleftharpoons}_X)$ *is a partial graph of* $(X, \rightleftharpoons_X)$.

In both cases, we have the following property.

Property 3 *Let* $x \in Y$ *(resp.* $x \in X$*). Let* $C = C^x_{\rightleftarrows}(Y) \in \mathcal{C}_{\rightleftarrows}[Y]$ *(resp.* $C = C^x_{\overset{\blacktriangle}{\rightleftarrows}}(X) \in \mathcal{C}_{\overset{\blacktriangle}{\rightleftarrows}}[X]$*), and* $C^x_{\rightleftarrows}(X) \in \mathcal{C}_{\rightleftarrows}[X]$ *be the unique connected components containing* x*. We have*

$$C \subseteq C^x_{\rightleftarrows}(X) \tag{1}$$

Moreover, for any $K \in \mathcal{C}_{\rightleftarrows}[X]$*, we have*

$$\big(K \cap C \neq \varnothing\big) \Rightarrow \big(K = C^x_{\rightleftarrows}(X)\big) \tag{2}$$

By progressively refining either X into successive subsets, or $\rightleftharpoons_X$ into successive (symmetric) subrelations, we can then build partition hierarchies defined as *trees*[4].

Property 4 *Let $(X_i)_{i=0}^k$ (resp. $(\rightleftharpoons_i)_{i=0}^k$) ($k \geqslant 0$) be such that $X_0 = X$ (resp. $\rightleftharpoons_0 = \rightleftharpoons_X$) and $X_{i+1} \subseteq X_i$ (resp. $\rightleftharpoons_{i+1} \subseteq \rightleftharpoons_i$) for all $i \in [\![0, k-1]\!]$. Let $(\bigcup_{i=0}^k C_{\rightleftarrows}[X_i], \sqsubseteq)$ (resp. $(\bigcup_{i=0}^k C_{\rightleftarrows_i}[X], \sqsubseteq)$) be the partially ordered multiset[5] where for all $K_\alpha \in C_{\rightleftarrows}[X_\alpha]$ (resp. $C_{\rightleftarrows_\alpha}[X]$) and $K_\beta \in C_{\rightleftarrows}[X_\beta]$ (resp. $C_{\rightleftarrows_\beta}[X]$) ($\alpha, \beta \in [\![0, k]\!]$), the order relation $\sqsubseteq$ extends $\subseteq$ as*

$$\big(K_\beta \sqsubseteq K_\alpha\big) \Leftrightarrow \Big(\big(K_\beta \subseteq K_\alpha\big) \wedge \big(\alpha \leqslant \beta\big)\Big) \tag{3}$$

For any $K, K' \in \bigcup_{i=0}^k C_{\rightleftarrows}[X_i]$ (resp. $\bigcup_{i=0}^k C_{\rightleftarrows_i}[X]$), we then have

$$\big(K \cap K' \neq \emptyset\big) \Rightarrow \big((K \sqsubseteq K') \vee (K' \sqsubseteq K)\big) \tag{4}$$

2.3 Partial and total partition hierarchies

Both ways to refine $(X, \rightleftharpoons_X)$ lead to partition hierarchies, but they differ with respect to the nature of these partitions. Indeed the structural refinement leads to total partitions, while the spatial refinement leads to partial partitions (as defined in [15]).

Property 5 *We have, for all $i \in [\![0, k-1]\!]$*

$$\begin{aligned} \bigcup C_{\rightleftarrows}[X_{i+1}] &\subseteq \bigcup C_{\rightleftarrows}[X_i] \\ \bigcup C_{\rightleftarrows_{i+1}}[X] &= \bigcup C_{\rightleftarrows_i}[X] \end{aligned} \tag{5}$$

Remark 6 *Typical examples of partial partition hierarchies are component-trees [19], where the successive $C_{\rightleftarrows}[X_i]$ are defined by considering the binary images obtained by thresholding a grey-level image $\mathcal{I} : X \to [\![0, k]\!]$. Typical examples of total partition hierarchies are (binary) partition-trees [18], where the successive $C_{\rightleftarrows_i}[X]$ are defined by progressively merging elementary parts of X, in a (multivalued) image $\mathcal{I} : X \to V$.*

3 Non-symmetry in adjacency: Semi-adjacency

In Sec. 2, a crucial hypothesis was the symmetry of the adjacency relation $\rightleftharpoons_X$ defined on X. We now investigate the effects induced by the relaxation of this hypothesis.

3.1 Semi-adjacency

Adjacency is defined as a relation being both irreflexive and symmetric. By relaxing the symmetry hypothesis, the obtained relation may (most of the time) no longer be an adjacency. We then introduce a more general notion to handle that case.

[4] Such trees are indeed *forests*, due to the non-necessary existence of a maximum (*i.e.*, a unique maximal element). For the sake of readibility, the term tree is used by abuse of notation.

[5] It may happen that successive partitions contain some similar connected components.

Definition 7 (Semi-adjacency) *Let $\rightharpoonup$ be an irreflexive binary relation on Ω. Such a relation is called a* semi-adjacency *on Ω. If $x \rightharpoonup y$, we say that x is* semi-adjacent *to y.*

We recall that, similarly to $\rightleftharpoons$, the relation $\rightharpoonup$ is still a subset of $\Omega \times \Omega$. However, by opposition to $\rightleftharpoons$, we have $(x \rightharpoonup y) \not\Rightarrow (y \rightharpoonup x)$.

3.2 Semi-connectedness

In the case of adjacency, the reflexive-transitive closure led to an equivalence relation that characterised the notion of connectedness. We follow here the same approach.

Definition 8 (Semi-connectedness) *Let $X \subseteq \Omega$ be a nonempty subset of Ω. We define the* semi-connectedness *relation $\rightarrow_X$ on X as the binary relation defined by the reflexive-transitive closure of the restriction $\rightharpoonup_X$ of $\rightharpoonup$ on X. If $x, y \in X$ satisfy $x \rightarrow_X y$, we say that x is* semi-connected *to y (in X).*

By opposition to $\rightleftarrows_X$, the relation $\rightarrow_X$ is not an equivalence relation, in general. (Note that a "semi-adjacency-based" notion of *stream* had also been considered in [3].)

Property 9 *The relation $\rightarrow_X$ is reflexive and transitive, but not necessarily symmetric.*

It is however possible to derive an equivalence relation from $\rightarrow_X$ by defining the strong connectedness relation $\circlearrowleft_X$ on X by

$$\big(x \circlearrowleft_X y\big) \Leftrightarrow \big((x \rightarrow_X y) \wedge (y \rightarrow_X x)\big) \tag{6}$$

If $x, y \in X$ satisfy $x \circlearrowleft y$, we say that x and y are strongly connected (in X). This notion of strong connectedness is classical the framework of (directed) graphs.

3.3 Semi-connected components

The notion of strong connectedness leads to equivalence classes of X, namely strongly connected components. The set of these strongly connected components is noted $\mathcal{C}_{\circlearrowleft}[X]$. Similarly, we can define the components that gather elements that are semi-connected.

Definition 10 (Semi-connected components) *Let $x \in X$. The* semi-connected component of X of basepoint x *is the subset of X defined by*

$$\mathcal{C}^x_{\rightarrow}(X) = \{y \in X \mid x \rightarrow_X y\} \tag{7}$$

The set of all the semi-connected components of X is noted $\mathcal{C}_{\rightarrow}[X]$.

By opposition to connected and strongly connected components, the semi-connected components of X do not necessarily form a partition of X.

Property 11 *$\mathcal{C}_{\rightarrow}[X]$ is a cover of X,* i.e.*, we have $\emptyset \notin \mathcal{C}_{\rightarrow}[X]$ and $X = \bigcup \mathcal{C}_{\rightarrow}[X]$.*

Indeed, it may happen that distinct semi-connected components have a nonempty intersection.

3.4 Links between semi-connected and strongly connected components

As stated by Def. 10, a semi-connected component is generated by a specific element, namely its basepoint. This basepoint is not necessarily unique: it may happen that $C^x_{\rightarrow}(X) = C^y_{\rightarrow}(X)$ for $x \neq y$. However, it is plain that such basepoints x and y are then strongly connected. From this fact, we straightforwardly derive the following property.

Property 12 *There exists an bijection between $\mathcal{C}_{\circlearrowleft}[X]$ and $\mathcal{C}_{\rightarrow}[X]$, expressed, for all $x, y \in X$, by*

$$\left(C^x_{\rightarrow}(X) = C^y_{\rightarrow}(X)\right) \Leftrightarrow \left(C^x_{\circlearrowleft}(X) = C^y_{\circlearrowleft}(X)\right) \tag{8}$$

Moreover, the partition $\mathcal{C}_{\circlearrowleft}[X]$ refines the cover $\mathcal{C}_{\rightarrow}[X]$.

Property 13 *Let $C \in \mathcal{C}_{\rightarrow}[X]$ be a semi-connected component. There exists a nonempty subset $\mathcal{P} \subseteq \mathcal{C}_{\circlearrowleft}[X]$ of strongly connected components such that $\mathcal{P}$ is a partition of C. In particular, $\mathcal{P}$ is defined as*

$$\mathcal{P} = \{C^x_{\circlearrowleft}[X] \mid x \in C\} \tag{9}$$

From the very definitions of $\rightarrow_X$ and $\circlearrowleft_X$, we finally derive the following property, that describes the structure of $\mathcal{C}_{\rightarrow}[X]$ induced by $\subseteq$, with respect to $\mathcal{C}_{\circlearrowleft}[X]$.

Property 14 *Let $C \in \mathcal{C}_{\rightarrow}[X]$ and $\mathcal{P} \subseteq \mathcal{C}_{\circlearrowleft}[X]$ be defined as above. Let $\mathcal{Q} \subseteq \mathcal{C}_{\rightarrow}[X]$ be the subset of semi-connected components defined by*

$$\mathcal{Q} = \{C^x_{\rightarrow}[X] \mid C^x_{\circlearrowleft}[X] \in \mathcal{P}\} = \{C^x_{\rightarrow}[X] \mid x \in C\} \tag{10}$$

The Hasse diagram of the partially ordered set $(\mathcal{Q}, \subseteq)$ is a directed acyclic graph (DAG), but not a tree in general. The maximum of $(\mathcal{Q}, \subseteq)$ is C, while its minimal elements belong to $\mathcal{P}$, and thus to $\mathcal{C}_{\circlearrowleft}[X]$.

4 Semi-connected components hierarchies

In this section, we still suppose that X is equipped with a semi-adjacency $\rightharpoonup_X$, which induces the semi-connectedness relation $\rightarrow_X$ and the strong connectedness relation $\circlearrowleft_X$.

4.1 Properties of semi-connected components hierarchies

Similarly to Sec. 2.2, we discuss here the effects of refining $(X, \rightharpoonup_X)$. Once again, this refinement can be done in two ways: (i) by defining $(Y, \rightharpoonup_Y)$ such that $Y \subseteq X$ and $\rightharpoonup_Y = \rightharpoonup_X \cap (Y \times Y)$ (Figs. 1, 3), or (ii), be defining $(X, \overset{\blacktriangle}{\rightharpoonup}_X)$ such that $\overset{\blacktriangle}{\rightharpoonup}_X \subseteq \rightharpoonup_X$.

Remark 15 *As in Rem. 2, $(Y, \rightharpoonup_Y)$ is a (directed) subgraph of $(X, \rightharpoonup_X)$, while $(X, \overset{\blacktriangle}{\rightharpoonup}_X)$ is a (directed) partial graph of $(X, \rightharpoonup_X)$.*

However, under the current hypotheses, the results of Prop. 3 are no longer totally valid. First, we have the following property that "extends" Eq. (1).

Property 16 *Let $x \in Y$ (resp. $x \in X$). Let $C = C^x_{\rightarrow}(Y) \in \mathcal{C}_{\rightarrow}[Y]$ (resp. $C = C^x_{\overset{\blacktriangle}{\rightarrow}}(X) \in \mathcal{C}_{\overset{\blacktriangle}{\rightarrow}}[X]$), and $C^x_{\rightarrow}(X) \in \mathcal{C}_{\rightarrow}[X]$ be the semi-connected components of basepoint x. We have*

$$C^x_{\rightarrow}(X) = \min_{\subseteq}\{K \in \mathcal{C}_{\rightarrow}[X] \mid C \subseteq K\} \tag{11}$$

Remark 17 *As in Prop. 3, this property guarantees that there exists an inclusion relation between the semi-connected components of same basepoint x between $\mathcal{C}_{\rightarrow}[Y]$ (resp. $\mathcal{C}_{\overset{\blacktriangle}{\rightarrow}}[X]$) and $\mathcal{C}_{\rightarrow}[X]$. However, contrarily to Prop. 3, it does not guarantee that for any given x, $C^x_{\rightarrow}(X)$ is the only semi-connected component that satisfies this inclusion relation. Nevertheless, it states that any other semi-connected component that has the same property also includes $C^x_{\rightarrow}(X)$.*

Still by comparison to Prop. 3, the analogue of Eq. (2) is now no longer satisfied.

Property 18 *With the same hypotheses as above, for any $K \in \mathcal{C}_{\rightarrow}[X]$, we have*

$$\big(K \cap C \neq \varnothing\big) \nRightarrow \big(K \supseteq C^x_{\rightarrow}(X)\big) \tag{12}$$

As a corollary, Prop. 4, cannot be generalised to the case of semi-connected components. Indeed, as stated by the following property, *semi-connected components hierachies are not organised as trees, but as DAGs* (Fig. 4(d)).

Property 19 *Let $(X_i)_{i=0}^k$ (resp. $(\rightharpoonup_i)_{i=0}^k$) ($k \geqslant 0$) be such that $X_0 = X$ (resp. $\rightharpoonup_0 = \rightharpoonup_X$) and $X_{i+1} \subseteq X_i$ (resp. $\rightharpoonup_{i+1} \subseteq \rightharpoonup_i$) for all $i \in [\![0, k-1]\!]$. Let us consider the partially ordered multiset $(\bigcup_{i=0}^k \mathcal{C}_{\rightarrow}[X_i], \sqsubseteq)$ (resp. $(\bigcup_{i=0}^k \mathcal{C}_{\rightarrow_i}[X], \sqsubseteq)$) (with $\sqsubseteq$ defined as in Eq. (3)). For any $K, K' \in \bigcup_{i=0}^k \mathcal{C}_{\rightarrow}[X_i]$ (resp. $\bigcup_{i=0}^k \mathcal{C}_{\rightarrow_i}[X]$), we have*

$$\big(K \cap K' \neq \varnothing\big) \nRightarrow \big((K \sqsubseteq K') \vee (K' \sqsubseteq K)\big) \tag{13}$$

Nevertheless, the intersection implies (under certain hypotheses) the inclusion.

Property 20 *Let $x \in Y$ (resp. $x \in X$), and $y \in X$. Let $C = C^x_{\rightarrow}(Y) \in \mathcal{C}_{\rightarrow}[Y]$ (resp. $C = C^x_{\overset{\blacktriangle}{\rightarrow}}(X) \in \mathcal{C}_{\overset{\blacktriangle}{\rightarrow}}[X]$). Let $C^y_{\rightarrow}(X) \in \mathcal{C}_{\rightarrow}[X]$. We have*

$$\big(x \in C \cap C^y_{\rightarrow}(X)\big) \Rightarrow \big(C \subseteq C^y_{\rightarrow}(X)\big) \tag{14}$$

4.2 Properties of strongly connected components hierarchies

Unlike semi-connected components, strongly connected components are defined as equivalence classes. Thus, they present common intrinsic properties with connected components (Fig. 2). In particular, Props. 3 and 4 can be extended to their case. (Note that $\mathcal{C}_{\overset{\blacktriangle}{\circlearrowleft}}[X], \mathcal{C}_{\circlearrowleft}[Y]$ are defined the same way as $\mathcal{C}_{\overset{\blacktriangle}{\rightleftarrows}}[X], \mathcal{C}_{\rightleftarrows}[Y]$ and $\mathcal{C}_{\overset{\blacktriangle}{\rightarrow}}[X], \mathcal{C}_{\rightarrow}[Y]$.)

Property 21 *Let $x \in Y$ (resp. $x \in X$). Let $C = C^x_{\circlearrowleft}(Y) \in \mathcal{C}_{\circlearrowleft}[Y]$ (resp. $C = C^x_{\overset{\blacktriangle}{\circlearrowleft}}(X) \in \mathcal{C}_{\overset{\blacktriangle}{\circlearrowleft}}[X]$) be the unique strongly connected component containing x. Then, there exists a unique $C^x_{\circlearrowleft}(X) \in \mathcal{C}_{\circlearrowleft}[X]$ that intersects (and actually includes) C.*

Property 22 *Let* $(X_i)_{i=0}^{k}$ *(resp.* $(\rightharpoonup_i)_{i=0}^{k}$*)* $(k \geqslant 0)$ *be such that* $X_0 = X$ *(resp.* $\rightharpoonup_0 = \rightharpoonup_X$*) and* $X_{i+1} \subseteq X_i$ *(resp.* $\rightharpoonup_{i+1} \subseteq \rightharpoonup_i$*) for all* $i \in [\![0, k-1]\!]$. *Let us consider the partially ordered multiset* $(\bigcup_{i=0}^{k} C_{\circlearrowleft}[X_i], \sqsubseteq)$ *(resp.* $(\bigcup_{i=0}^{k} C_{\circlearrowleft_i}[X], \sqsubseteq)$*) (with* $\sqsubseteq$ *defined as in Eq. (3)). For any two strongly connected components* K, K' *of this set, we have*

$$\big(K \cap K' \neq \varnothing\big) \Rightarrow \big((K \sqsubseteq K') \vee (K' \sqsubseteq K)\big) \tag{15}$$

Then, by progressively refining X into successive subsets, or $\rightharpoonup_X$ into successive subrelations, we can build strongly connected components hierarchies as trees (Fig. 4(a)).

4.3 Semi-connected components hierarchies as enriched strongly connected components hierarchies

On one hand, it has been observed in Sec. 4.1, that the semi-connected components hierarchies induced by progressively refining $(X, \rightharpoonup_X)$, have a structure which cannot be trivially handled. Indeed, these (cover) hierarchies are DAGs, due to inclusions or intersections (without inclusion) between different components at same levels.

On the other hand, it has been observed in Sec. 4.2, that the strongly connected components hierarchies, induced by the very same process, have a much simpler structure. Indeed, these (partition) hierarchies are trees.

Based on the bijection (Prop. 12) that exists between semi-connected and strongly connected components, it is however possible to model the (DAG) hierarchy of semi-connected components as a (tree) hierarchy of the associated strongly connected components, enriched at each level by a “local” DAG that represents the inclusion relation between the semi-connected components of this level. This model is formally expressed by the following proposition.

Proposition 23 *Let* $\alpha, \beta \in [\![0, k]\!]$, *with* $\alpha \leqslant \beta$. *Let* $x, y \in X$. *Let* $C^x_{\rightarrow}(X_\alpha), C^y_{\rightarrow}(X_\beta) \in \bigcup_{i=0}^{k} C_{\rightarrow}[X_i]$ *(resp.* $C^x_{\rightarrow_\alpha}(X), C^y_{\rightarrow_\beta}(X) \in \bigcup_{i=0}^{k} C_{\rightarrow_i}[X]$*). Then, we have*

$$\begin{gathered}\big(C^y_{\circlearrowleft}(X_\beta) \subseteq C^x_{\circlearrowleft}(X_\alpha)\big) \Rightarrow \big(C^y_{\rightarrow}(X_\beta) \sqsubseteq C^x_{\rightarrow}(X_\alpha)\big) \\ \textit{(resp. } \big(C^y_{\circlearrowleft_\beta}(X) \subseteq C^x_{\circlearrowleft_\alpha}(X)\big) \Rightarrow \big(C^y_{\rightarrow_\beta}(X) \sqsubseteq C^x_{\rightarrow_\alpha}(X)\big) \textit{)}\end{gathered} \tag{16}$$

and

$$\begin{gathered}\big(C^y_{\rightarrow}(X_\beta) \sqsubseteq C^x_{\rightarrow}(X_\alpha)\big) \Rightarrow \Big(\big(C^y_{\circlearrowleft}(X_\beta) \subseteq C^y_{\circlearrowleft}(X_\alpha)\big) \wedge \big(C^y_{\rightarrow}(X_\alpha) \subseteq C^x_{\rightarrow}(X_\alpha)\big)\Big) \\ \textit{(resp. } \big(C^y_{\rightarrow_\beta}(X) \sqsubseteq C^x_{\rightarrow_\alpha}(X)\big) \Rightarrow \Big(\big(C^y_{\circlearrowleft_\beta}(X) \subseteq C^y_{\circlearrowleft_\alpha}(X)\big) \wedge \big(C^y_{\rightarrow_\alpha}(X) \subseteq C^x_{\rightarrow_\alpha}(X)\big)\Big) \textit{)}\end{gathered} \tag{17}$$

Proof Let us suppose that $C^y_{\circlearrowleft}(X_\beta) \subseteq C^x_{\circlearrowleft}(X_\alpha)$ (resp. $C^y_{\circlearrowleft_\beta}(X) \subseteq C^x_{\circlearrowleft_\alpha}(X)$). Then, we have $y \in C^x_{\circlearrowleft}(X_\alpha)$ (resp. $y \in C^x_{\circlearrowleft_\alpha}(X)$), and thus $C^y_{\circlearrowleft}(X_\alpha) = C^x_{\circlearrowleft}(X_\alpha)$ (resp. $C^y_{\circlearrowleft_\alpha}(X) = C^x_{\circlearrowleft_\alpha}(X)$). From Eq. (8) (Prop. 12), we then have $C^y_{\rightarrow}(X_\alpha) = C^x_{\rightarrow}(X_\alpha)$ (resp. $C^y_{\rightarrow_\alpha}(X) = C^x_{\rightarrow_\alpha}(X)$). From the definition of X_α and X_β (resp. $\rightharpoonup_\alpha$ and $\rightharpoonup_\beta$) (Prop. 4), and the fact that $\alpha \leqslant \beta$, we straightforwardly have $C^y_{\rightarrow}(X_\beta) \subseteq C^y_{\rightarrow}(X_\alpha)$ (resp. $C^y_{\rightarrow_\beta}(X) \subseteq C^y_{\rightarrow_\alpha}(X)$), and then $C^y_{\rightharpoonup}(X_\beta) \subseteq C^x_{\rightharpoonup}(X_\alpha)$ (resp. $C^y_{\rightharpoonup_\beta}(X) \subseteq C^x_{\rightharpoonup_\alpha}(X)$). Eq. (16) then follows from the definition of $\sqsubseteq$, provided in Eq. (3) (Prop. 4).

Let us now suppose that $C^y_{\rightharpoonup}(X_\beta) \sqsubseteq C^x_{\rightharpoonup}(X_\alpha)$ (resp. $C^y_{\rightharpoonup_\beta}(X) \sqsubseteq C^x_{\rightharpoonup_\alpha}(X)$). From the definition of $\sqsubseteq$, we then have $C^y_{\rightharpoonup}(X_\beta) \subseteq C^x_{\rightharpoonup}(X_\alpha)$ (resp. $C^y_{\rightharpoonup_\beta}(X) \subseteq C^x_{\rightharpoonup_\alpha}(X)$). Let us consider $C^y_{\rightharpoonup}(X_\alpha)$ (resp. $C^y_{\rightharpoonup_\alpha}(X)$); this set exists since $y \in C^y_{\rightharpoonup}(X_\beta) \subseteq C^x_{\rightharpoonup}(X_\alpha) \subseteq X_\alpha$ (resp. $y \in C^y_{\rightharpoonup_\beta}(X) \subseteq C^x_{\rightharpoonup_\alpha}(X) \subseteq X$). From the very definition of semi-connectedness (Def. 10), $y \in C^x_{\rightharpoonup}(X_\alpha)$ (resp. $y \in C^x_{\rightharpoonup_\alpha}(X)$) implies that $C^y_{\rightharpoonup}(X_\alpha) \subseteq C^x_{\rightharpoonup}(X_\alpha)$ (resp. $C^y_{\rightharpoonup_\alpha}(X) \subseteq C^x_{\rightharpoonup_\alpha}(X)$), while $X_\beta \subseteq X_\alpha$ (resp. $\rightharpoonup_\beta \subseteq \rightharpoonup_\alpha$) implies that $C^y_{\rightharpoonup}(X_\beta) \subseteq C^y_{\rightharpoonup}(X_\alpha)$ (resp. $C^y_{\rightharpoonup_\beta}(X) \subseteq C^y_{\rightharpoonup_\alpha}(X)$). Finally, we derive from Eq. (8) (Prop. 12) that $C^y_{\circlearrowleft}(X_\beta) \subseteq C^y_{\circlearrowleft}(X_\alpha)$ (resp. $C^y_{\circlearrowleft_\beta}(X) \subseteq C^y_{\circlearrowleft_\alpha}(X)$), and Eq. (17) then follows. ∎

Remark 24 *Practically, this proposition proves that the standard relation $\sqsubseteq$, that models the inclusion relation between the semi-connected components at distinct levels of a hierarchy (Fig. 4(d)), can be conveniently handled by simultaneously using the inclusion relation between the strongly connected components associated to the semi-connected ones, and the inclusion relation between the semi-connected components of same level in the hierarchy (Fig. 4(c)). This representation, which is formalised in Diagram* (18)*, has the two following virtues: (i) it is information lossless, with respect to $\sqsubseteq$; and (ii) it replaces the Hasse diagram of $\sqsubseteq$, which is a complex DAG, by a tree structure that is enriched by "local" simple DAGs at each level of the tree. Moreover, its complexity is not excessive by comparison to the Hasse diagram of $\sqsubseteq$.*

$$
\begin{tikzcd}
C^x_{\rightharpoonup}(X_\alpha) & C^y_{\rightharpoonup}(X_\alpha) \arrow[l, "\sqsubseteq"'] \arrow[r, leftrightarrow, "\sim"] & C^y_{\circlearrowleft}(X_\alpha) & C^x_{\rightharpoonup_\alpha}(X) & C^y_{\rightharpoonup_\alpha}(X) \arrow[l, "\sqsubseteq"'] \arrow[r, leftrightarrow, "\sim"] & C^y_{\circlearrowleft_\alpha}(X) \\
C^y_{\rightharpoonup}(X_\beta) \arrow[u, "\sqsubseteq"'] \arrow[rr, leftrightarrow, "\sim"] & & C^y_{\circlearrowleft}(X_\beta) \arrow[u, "\sqsubseteq"'] & C^y_{\rightharpoonup_\beta}(X) \arrow[u, "\sqsubseteq"'] \arrow[rr, leftrightarrow, "\sim"] & & C^y_{\circlearrowleft_\beta}(X) \arrow[u, "\sqsubseteq"']
\end{tikzcd}
\tag{18}
$$

5 Extending standard tree structures

The "enriched trees" defined above are compliant with standard partition hierarchies when the considered semi-adjacency is indeed an adjacency.

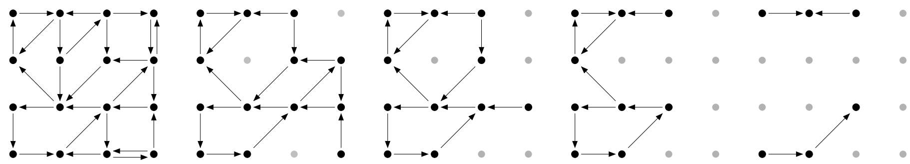
Property 25 *If $\rightharpoonup_X$ is symmetric, we have $C_{\rightharpoonup}[X] = C_{\circlearrowleft}[X]$ ($= C_{\leftrightarrows}[X]$ by considering $\rightharpoonup_X$ as an adjacency relation). In such conditions, the Hasse diagram induced by $\sqsubseteq$, and the associated enriched tree are equal and both have a tree structure.*

In a reverse way, we show how some standard (total and partial) partition hierarchies can be generalised to handle semi-connected components hierarchies.

5.1 Partial partition hierarchies and component-trees

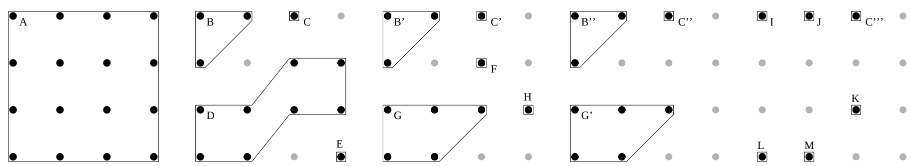
We consider the spatial way to refine $(X, \rightharpoonup_X)$. With this *modus operandi*, the semi-connected components, present in successive subsets, have the same lower elements.

Property 26 *Let $K \in C_{\rightharpoonup}[X] \cap C_{\rightharpoonup}[Y]$. Then we have $\{K' \in C_{\rightharpoonup}[X] \mid K' \subseteq K\} = \{K' \in C_{\rightharpoonup}[Y] \mid K' \subseteq K\}$.*



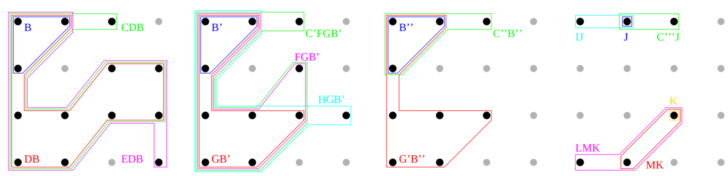
(a) $(X_0, \rightharpoonup_{X_0})$ (b) $(X_1, \rightharpoonup_{X_1})$ (c) $(X_2, \rightharpoonup_{X_2})$ (d) $(X_3, \rightharpoonup_{X_3})$ (e) $(X_4, \rightharpoonup_{X_4})$

Fig. 1. Subsets X_i of a set X, equipped with subrelations $\rightharpoonup_{X_i}$ of a semi-adjacency relation $\rightharpoonup$, for $i = 0$ to 4, with $X_{i+1} \subseteq X_i$ for all $i \in [\![0, 3]\!]$, and $\rightharpoonup_{X_i} = \rightharpoonup \cap (X_i \times X_i)$ for all $i \in [\![0, 4]\!]$.



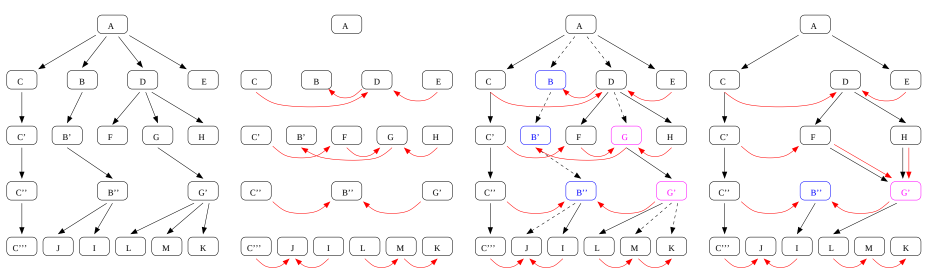
(a) $C_{\circlearrowleft}[X_0]$ (b) $C_{\circlearrowleft}[X_1]$ (c) $C_{\circlearrowleft}[X_2]$ (d) $C_{\circlearrowleft}[X_3]$ (e) $C_{\circlearrowleft}[X_4]$

Fig. 2. The strongly connected components of X_i. Each one is labeled by a capital letter: A, B, C, etc. When a component Z appears in several $C_{\circlearrowleft}[X_i]$, it is successively labeled as Z, Z', Z'', etc.



(a) $C_{\rightharpoonup}[X_0]$ (b) $C_{\rightharpoonup}[X_1]$ (c) $C_{\rightharpoonup}[X_2]$ (d) $C_{\rightharpoonup}[X_3]$ (e) $C_{\rightharpoonup}[X_4]$

Fig. 3. The semi-connected components of X_i. Each one is labeled by the capital letters corresponding to the strongly connected components that form its partition: A, CDB, C'FGB', etc.



(a) $(\bigcup_{i=0}^{4} C_{\circlearrowleft}[X_i], \sqsubseteq)$ (b) $(C_{\rightharpoonup}[X_i], \subseteq)_{i=0}^{4}$ (c) "(a) $\cup$ (b)" (d) $(\bigcup_{i=0}^{4} C_{\rightharpoonup}[X_i], \sqsubseteq)$

Fig. 4. (a) Hasse diagram for $\sqsubseteq$ of all the $C_{\circlearrowleft}[X_i]$. (b) Hasse diagrams for $\subseteq$ of each $C_{\rightharpoonup}[X_i]$. (c) Proposed structure (enriched tree), *i.e.*, the fusion of (a) and (b). (d) Hasse diagram for $\sqsubseteq$ (DAG) of all the $C_{\rightharpoonup}[X_i]$, modeled by (c). The dashed arrows in (c) are "extra" links with respect to (d). The blue and magenta nodes are elements that may be collapsed in (c) without loss of information. (b–d) For the sake of readability, the nodes are labeled by their first letter, with respect to Fig. 3.

Two examples of such semi-connected components are provided in Fig. 4(c). These (sets of) components, namely B, B' and B" (resp. GB' and G'B"), can be unified into a single semi-connected component B" (resp. G'B"), as it is already trivially done in Fig. 4(d) (where B, B', B" and GB' and G'B" would form two chains, respectively, otherwise).

Remark 27 *Prop. 26 implies that the partially ordered multiset* $(\bigcup_{i=0}^{k} C_{\rightarrow}[X_i], \sqsubseteq)$ *can be handled as a partially ordered* set*. However, it is not sufficient to claim that there exists an equivalence between the set* $\bigcup_{i=0}^{k} C_{\rightarrow}[X_i]$ *of all the semi-connected components, and the set* $\bigcup_{i=0}^{k} C_{\circlearrowleft}[X_i]$ *of all the strongly connected components. In particular, the semi-connected components CDB, C'FGB', C"B" and C'"J of Fig. 4 straightforwardly provide a counter-example to that claim.*

From this remark, we can derive that it is possible to extend the notion of component-tree [19, 8], but only up to an equivalence between equal strongly connected components that provide basepoints for semi-connected components.

5.2 Total partition hierarchies and hierarchical connectivities

We now consider the structural way to refine $(X, \rightharpoonup_X)$.

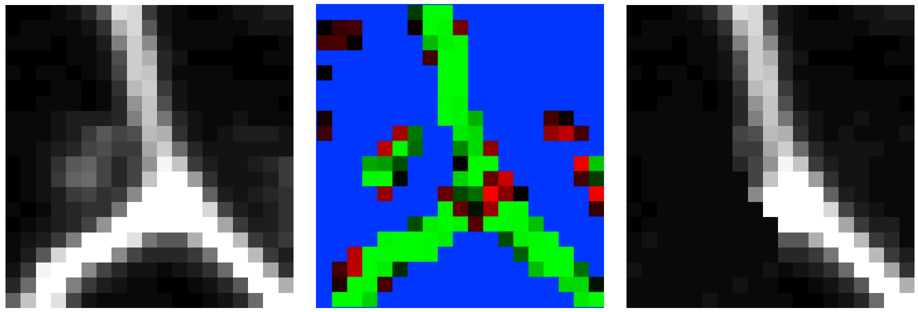
Remark 28 *By following this* modus operandi*, the semi-connected components that are present in successive subsets do not necessarily have the same lower elements. This difference with Rem. 27 derives from the fact that the semi-connected components of* $C_{\rightarrow_i}[X]$ *are composed of elements of X, but are actually defined with respect to* $\rightarrow_i$*.*

Practically it is possible to extend (still up to an equivalence) some notions of partition-tree, *e.g.*, those modeling hierarchical connectivities in fuzzy paradigms [17, 1].

6 Application example

Basically, the enriched tree, that models a semi-connected components hierarchy, can be built in two steps. The first step consists of computing the hierarchy of the strongly connected components. This can be done by considering an approach based on Tarjan's algorithm. The second step consists of computing, at each level of the tree, the links derived from the inclusion between the semi-connected components. They actually correspond to the remaining semi-adjacency links between the strongly connected components that model the semi-connected ones. A complete algorithmic discussion is beyond the scope of this article, and will be developed in details in further works.

We finally propose a (simple) example, which purpose is to illustrate the relevance of the notion of semi-connectedness, and its methodological usefulness. Let us consider a digital grey-level image $\mathcal{I} : X \to V$ (Fig. 5(a)), that visualises neurites. A Hessian filter can be applied on $\mathcal{I}$, to classify the pixels as linear (L, in green), blobs (B, in red), and others (O, in blue) (Fig. 5(b)). From this classification, the standard 4-adjacency relation defined on $X \subseteq \mathbb{Z}^2$ can be restricted to a semi-adjacency relation $\rightharpoonup_X$ defined by $x \rightharpoonup_X y$ iff x and y belong to the same class, or $x \in O$ or B while $y \in L$.



(a) Image $\mathcal{I} : X \to V$ (b) Classification $\{L, B, O\}$ (c) Filtered image

Fig. 5. Neurite filtering from semi-connected components hierarchy (see text).

From the semi-connected components hierarchy of $\mathcal{I}$, with respect to $\rightharpoonup_X$ (which is an extension of a component-tree), we can filter the semi-connected components presenting a linear shape. For instance, in Fig. 5(c), two of the three linear patterns have been preserved, while a third one has been removed due to its orientation.

Of course, this example is of limited interest, since the crisp classification of X into three classes strongly constrains the space of the possible results. A more satisfactory solution may be to perform a fuzzy classification, that would lead to define semi-adjacencies not only with respect to the level sets of I, but also to the level sets of the fuzzy classification scores. Such perspective works mat be relevantly considered in the framework of hypertrees [12]. Beyond these considerations, one may notice that with a standard component-tree, the three linear patterns would have been merged in the same connected component, forbiding (i) linearity characterisation, and (ii) splitting of the two linear patterns of highest intensity. The proposed example, despite its simplicity, then clearly illustrates the potential usefulness of semi-connected filtering.

7 Conclusion

This work provides first results which demonstrate that cover hierarchies derived from semi-adjacency (i) can be handled via (enriched) tree structures; (ii) provide a way to generalise classical structures such as component-trees and partition-trees; and (iii) may be involved in image processing tasks, in the framework of connected operators.

From a theoretical point of view, the links that may exist between such hierarchies and those induced by hyperconnections [14] could be explored. The relationships with other non-tree hierarchies [13] could also be investigated. More generally, the notion of semi-connection could be axiomatically formalised, beyond the only framework of graphs, similarly to the proposal of Serra [21] for connections.

From a methodological point of view, the extension of segmentation paradigms based on optimal tree-cuts [6, 23] could be considered, with challenges related to algorithmic complexities. New operators could also be designed to provide "disconnection" filters, that may be seen as dual operators with respect to reconnection filters [5].

References

1. I. Bloch, H. Maître, and M. Anvar. Fuzzy adjacency between image objects. *Int. J. Uncertain. Fuzz.*, 5:615–654, 1997.
2. U. Braga-Neto and J. K. Goutsias. A theoretical tour of connectivity in image processing and analysis. *J. Math. Imaging Vis.*, 19:5–31, 2003.
3. J. Cousty, G. Bertrand, L. Najman, and M. Couprie. Watershed cuts: Minimum spanning forests and the drop of water principle. *IEEE T. Pattern Anal.*, 31:1362–1374, 2009.
4. J. Cousty, G. Bertrand, L. Najman, and M. Couprie. Watershed cuts: Thinnings, shortest path forests, and topological watersheds. *IEEE T. Pattern Anal.*, 32:925–939, 2010.
5. A. Dufour, O. Tankyevych, B. Naegel, H. Talbot, C. Ronse, J. Baruthio, P. Dokládal, and N. Passat. Filtering and segmentation of 3D angiographic data: Advances based on mathematical morphology. *Med. Image Anal.*, In Press. doi: `10.1016/j.media.2012.08.004`.
6. L. Guigues, J.-P. Cocquerez, and H. Le Men. Scale-sets image analysis. *Int. J. Comput. Vision*, 68:289–317, 2006.
7. H. J. A. M. Heijmans. Connected morphological operators for binary images. *Comput. Vis. Image Und.*, 73:99–120, 1999.
8. R. Jones. Connected filtering and segmentation using component trees. *Comput. Vis. Image Und.*, 75:215–228, 1999.
9. F. Meyer and L. Najman. Segmentation, minimum spanning tree and hierarchies. In L. Najman and H. Talbot, editors, *Mathematical Morphology: From Theory to Applications*, pages 229–261. ISTE/J. Wiley & Sons, 2010.
10. L. Najman. On the equivalence between hierarchical segmentations and ultrametric watersheds. *J. Math. Imaging Vis.*, 40:231–247, 2011.
11. G. K. Ouzounis and M. H. F. Wilkinson. Mask-based second-generation connectivity and attribute filters. *IEEE T. Pattern Anal.*, 29:990–1004, 2007.
12. N. Passat and B. Naegel. Component-hypertrees for image segmentation. In *ISMM*, volume 6671 of *LNCS*, pages 284–295. Springer, 2011.
13. N. Passat and B. Naegel. Component-trees and multivalued images: Structural properties. Technical report, INRIA-00611714, 2012. url: `http://hal.inria.fr/inria-00611714`.
14. B. Perret, S. Lefèvre, C. Collet, and E. Slezak. Hyperconnections and hierarchical representations for grayscale and multiband image processing. *IEEE T. Image Proc.*, 21:14–27, 2012.
15. C. Ronse. Set-theoretical algebraic approaches to connectivity in continuous or digital spaces. *J. Math. Imaging Vis.*, 8:41–58, 1998.
16. A. Rosenfeld. Connectivity in digital pictures. *J. Assoc. Comput. Mach.*, 17:146–160, 1970.
17. A. Rosenfeld. Fuzzy digital topology. *Inform. Control*, 40:76–87, 1979.
18. P. Salembier and L. Garrido. Binary partition tree as an efficient representation for image processing, segmentation and information retrieval. *IEEE T. Image Proc.*, 9:561–576, 2000.
19. P. Salembier, A. Oliveras, and L. Garrido. Anti-extensive connected operators for image and sequence processing. *IEEE T. Image Proc.*, 7:555–570, 1998.
20. P. Salembier and M. H. F. Wilkinson. Connected operators: A review of region-based morphological image processing techniques. *IEEE Signal Proc. Mag.*, 26:136–157, 2009.
21. J. Serra. Mathematical morphology for Boolean lattices. In J. Serra, editor, *Image Analysis and Mathematical Morphology, II: Theoretical Advances*, pages 37–58. Academic Press, London, 1988.
22. J. Serra. Connectivity on complete lattices. *J. Math. Imaging Vis.*, 9:231–251, 1998.
23. J. Serra. Tutorial on connective morphology. *IEEE J. Sel. Top. Signa.*, 6:739–752, 2012.
24. P. Soille. Constrained connectivity for hierarchical image partitioning and simplification. *IEEE T. Pattern Anal.*, 30:1132–1145, 2008.