Robust controller design for uncertain T-S fuzzy systems with time-varying delays

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***Abstract*— This paper analyzes the robust control problems for a class of uncertain Takagi-Sugeno (T-S) fuzzy systems with time varying delays. T-S fuzzy models are employed to represent uncertain delayed nonlinear systems. A Parallel Distributed Compensation (PDC) control law, including both memoryless and delayed state feedback, is considered for stabilization purpose. Based on the choice of a convenient Lyapunov-Krasovskii Functional (LKF) and introducing free weighting matrices, sufficient delay-dependent controller design conditions are derived in terms of linear matrix inequality (LMI). Finally, a numerical example is presented to demonstrate the effectiveness of the proposed approach and the conservatism improvement regarding to previous results.**

I. INTRODUCTION

An efficient way to represent a non-linear system consists on a representation in the form of Takagi-Sugeno fuzzy model [1]. These ones have been extensively investigated due to their effectiveness and powerful tool to deal with several nonlinear control problems, see e.g. [2], [3], [4], [5], [6]. Historically, T-S fuzzy models can be interpreted as a set of affine linear models interconnected by nonlinear membership functions. The sub-models express the dynamics of a system around particular operating points of the state space. Moreover, using the well-known sector nonlinearity approach [2], a systematic way to obtain the membership functions is available and a T-S model matches exactly the considered nonlinear one in a compact set of its state space.

The stability problem for T-S fuzzy models are often studied from the well-known Lyapunov direct method. For stabilization issues, the parallel distributed compensation (PDC) control scheme can be considered and the design conditions are often given as Linear Matrix Inequalities (LMI) [2], [3], [7]. However, these conditions being only sufficient, many efforts are done to reduce their conservatism, see e.g. [8], [9], [10].

In practice, time-delays are encountered in a wide range of engineering control processes such like chemical or metallurgical ones, network systems, pneumatic or hydraulic plants, telecommunications, and so on [11]. Note that, in such practical applications, time-delays cannot be neglected since they are sources of instability, oscillations or degraded performances for the considered dynamics [12]. Thus, it appears necessary to develop tools dedicated to delayed systems such as stability and stabilization criteria. In this context, several studies have been published in this latest years, see for example [13], [14], [15], [16], [17] and the references therein. Likewise, the stability of time delay systems may vanish due to structural uncertainties [18], [19], [20]. Thus, designing robust controllers for uncertain time-delayed T-S systems remain a challenging problem.

Based on the sizes of time-delays, the stability criteria of delayed T-S fuzzy systems can be classified into two category: delay-independent ones and delay-dependent ones. Delay-independent criteria have been proposed in [12], [13], [21], which provides stability conditions which are independent of the size of the delays i.e., they do not include information related to delays. In general, this lack of information leads to conservative results. Hence, delay-dependent criteria have been proposed [14], [15], [17], [22], [23], [24]. These include informations regarding to the delays such as their maximal values or the bounds of their derivatives. Such approaches are less conservative than delay-independent ones and are relevant when the delays are time varying.

In this paper, the problem of robust PDC controller design for uncertain T-S fuzzy systems with time-varying delay is considered as an extension to our preliminary work [25]. In this context, following the works done in [23], [16], [26], the considered controller includes memoryless and delayed state feedback to stabilize a uncertain T-S fuzzy model. Therefore, one of the improvement proposed in the present study consists in a convenient choice of a Lyapunov-Krasovksii functional (LKF). Indeed, LKF allows to exploit the information of the delays and their derivatives to derive the stability conditions. To improved the conservatism, the considered LKF is based on a state vector extension. Moreover, following the works of [27], [28], [22], [29], to provide more degree of freedom to the convex optimization problem, free-weighting matrices are introduced into the LMI conditions. Finally, the benefit of the proposed LMI conditions in terms of conservatism, regarding to several previous results, is illustrated through an academic example.

II. PRELIMINARIES

Let us consider an uncertain T-S fuzzy system with time-delays with r fuzzy rules given by:

$Rule\, i$: IF $z_1(t)$ is μ_{i1} and ... and $z_p(t)$ is μ_{ip} THEN

$$\begin{cases} \dot{x}(t) = & (A_i + \Delta A_i)\, x(t) + \left(A_i^d + \Delta A_i^d\right) x\left(t - \tau(t)\right) \\ & + (B_i + \Delta B_i)\, u(t) \\ x(t) = & \phi(t),\ t \in \left[-\bar{\tau}, 0\right], i = 1, ..., r \end{cases} \tag{1}$$

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where p is the number of fuzzy sets μ_{ij} in the i^{th} rule (i.e. $j=1,...,p$), $z(t)=\left[\begin{array}{ccc} z_1(t) & ... & z_p(t)\end{array}\right]\in\mathbb{R}^P$ is a known vector of premise variables which may depend on the state vector $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$ is the control input vector, A_i, A_i^d and B_i are known constant matrices with compatible dimensions, $\tau(t)\in[0,\bar{\tau}]$ represent a time-varying delay with $\bar{\tau}<+\infty$ and $\dot{\tau}(t)\leq\eta<+\infty$ ($\bar{\tau}$ and η are constant scalars), $\phi(t)$ is a vector-valued initial function for $t\in[-\bar{\tau},0]$, ΔA_i , ΔA_i^d and ΔB_i are unknown matrices representing Lebesgue measurable uncertainties, which can be rewritten as [30]:

$$\Delta A_i = H_i\delta(t)E_{ai} \tag{2}$$

$$\Delta A_i^d = H_i\delta(t)E_{ai}^d \tag{3}$$

$$\Delta B_i = H_i\delta(t)E_{bi} \tag{4}$$

where H_i, E_{ai}, E_{ai}^d and E_{bi} are known constant real matrices with compatible dimensions, $\delta(t)$ is an unknown real time-varying vector satisfying:

$$\delta^T(t)\delta(t)\leq I \tag{5}$$

Notations: In the sequel, for simplifying the notations, a star (*) in a matrix denotes a block transpose quantity. Let us denote $\bar{A}_i=A_i+\Delta A_i$, $\bar{A}_i^d=A_i^d+\Delta A_i^d$ and $\bar{B}_i=B_i+\Delta B_i$. Moreover, for any set of matrices M_i of appropriate dimensions, one denotes $M_h=\sum_{i=1}^r h_i(z(t))M_i$ and $M_{hh}=\sum_{i=1}^r\sum_{j=1}^r h_i(z(t))h_j(z(t))M_{ij}$.

Let $\mu_{ij}(z_j(t))\in[0,1]$ be the grade of membership of $z_j(t)$ in μ_{ij}. By using the center-average defuzzification, product inference and singleton fuzzifier, one may define the membership functions as:

$$h_i(z(t))=\frac{w_i(z(t))}{\sum_{i=1}^r w_i(z(t))} \tag{6}$$

with $w_i(z(t))=\prod_{i=1}^r\mu_{ij}(z(t))$.

Hence the membership functions $h_i(t)$ hold the convex sum properties, i.e. for $i=1,...,r$, $h_i(z(t))\geq 0$ and $\sum_{i=1}^r h_i(z(t))=1$.

From (6), a compact representation of the uncertain T-S fuzzy model (1) is be given by:

$$\dot{x}(t)=\sum_{i=1}^r h_i(z(t))\left(\bar{A}_i x(t)+\bar{A}_i^d x(t-\tau(t))+\bar{B}_i u(t)\right) \tag{7}$$

or with the above defined notations:

$$\dot{x}(t)=\bar{A}_h x(t)+\bar{A}_h^d x\left(t-\tau(t)\right)+\bar{B}_h u(t) \tag{8}$$

To stabilize the uncertain T-S fuzzy models (8) and due to the existence of delayed terms, one considers the following PDC control law [23]:

$$u(t)=u_m(t)+u_d(t) \tag{9}$$

with $u_m(t)=\sum_{j=1}^r h_j(z(t))K_j x(t)$ and $u_d(t)=\sum_{j=1}^r h_j(z(t))K_j^d x(t-\tau(t))$, and where K_j and K_j^d , for $j=1,...,r$, are the controller gain matrices to be designed.

Assumption 1: When not explicitly stated, the time-varying delay $\tau(t)$ is assumed to be available online at any time t.

Note that the control law (9) requires assumption 1, which is considered as a general case to derive new design conditions. Then, it will be shown that straightforward simplifications may apply for particular cases such like constant delays.

From (8) and (9), the closed-loop dynamics may be written as:

$$\dot{x}(t)=\left(\bar{A}_h+\bar{B}_h K_h\right)x(t)+\left(\bar{A}_h^d+\bar{B}_h K_h^d\right)x\left(t-\tau(t)\right) \tag{10}$$

The purpose of this paper is to propose LMI conditions for the design of a PDC controller (9) such that the closed-loop system (10) is globally asymptotically stable (GAS). To achieve this goal and to provide relaxed conditions the following lemmas will be used.

Lemma 1: [22] For any constant matrices $Q_{11}=Q_{11}^T$, $Q_{22}=Q_{22}^T$, and $Q_{12}\in\mathbb{R}^{n\times n}$ satisfying $\left[\begin{array}{cc} Q_{11} & Q_{12}\\ * & Q_{22}\end{array}\right]\geq 0$, a positive scalar function $\tau(t)\leq\bar{\tau}<+\infty$, and a vector function $\dot{x}(t):[-\bar{\tau},0]\rightarrow\mathbb{R}^n$, such that the following integrations are well defined, then:

$$\begin{array}{l} -\bar{\tau}\int_{t-\bar{\tau}}^t\left[\begin{array}{c} x(s)\\ \dot{x}(t)\end{array}\right]^T\left[\begin{array}{cc} Q_{11} & Q_{12}\\ * & Q_{22}\end{array}\right]\left[\begin{array}{c} x(s)\\ \dot{x}(s)\end{array}\right]ds\\ \quad\leq\theta^T(t)\left[\begin{array}{ccc} -Q_{22} & Q_{22} & -Q_{12}^T\\ Q_{22} & -Q_{22} & Q_{12}^T\\ -Q_{12} & Q_{12} & -Q_{11}\end{array}\right]\theta(t)\end{array} \tag{11}$$

with $\theta(t)=\left[\begin{array}{c} x(t)\\ x(t-\tau(t))\\ \int_{t-\tau(t)}^t x(s)ds\end{array}\right]$.

Lemma 1 is derived from the Jensen's integral inequality [28]. It will be used in the proof of the theorem proposed in the next section.

Lemma 2: [8] For $(i,j)\in\{1,...,r\}^2$, Let $\mathit{\Gamma}_{ij}$ be matrices of appropriate dimensions. $\mathit{\Gamma}_{hh}<0$ is satisfied if both the following conditions hold:

$$\begin{cases}\Gamma_{ii}<0,\forall i\in\{1,...,r\}\\ \frac{2}{r-1}\Gamma_{ii}+\mathit{\Gamma}_{ij}+\Gamma_{ji}<0,\forall\,(i,j)\in\{1,...,r\}_{i\neq j}^2\end{cases} \tag{12}$$

Lemma 2 is considered for conservatism reduction [8]. Note that, among relaxation lemmas, it constitutes a good compromise between complexity and computational burden [9].

To conclude this section, the following lemma will be employed to deal with uncertainties in the proof of the theorem proposed in the next section.

Lemma 3: [30] Let $Q=Q^T$, H, E and be real matrices of appropriate dimensions and a vector $\delta(t)$ satisfying (3). The inequality:

$$Q+H\delta(t)E+E^T\delta^T(t)H^T\leq 0 \tag{13}$$

is satisfied if there exists a scalar $\lambda>0$ such that:

$$Q+\lambda HH^T+\lambda^{-1}E^TE\leq 0 \tag{14}$$

III. Main result

In this section, the goal is to provide new LMI-based delay dependent conditions for the design of PDC controllers (9) which globally asymptotically stabilize uncertain T-S fuzzy models with time varying delays (8). The main result is given by the following theorem.

Theorem 1: For given scalars $\bar{\tau} > 0$ and $\eta > 0$, the uncertain T-S fuzzy model with time varying delays (8) is globally asymptotically stabilized by the robust PDC controller (9) if, for any time-delay $\tau(t) \in [0, \bar{\tau}]$ with $\dot{\tau}(t) \leq \eta$, and for all $(i,j) \in \{1,...,r\}^2$, there exist real matrices with appropriate dimensions $L = L^T > 0$, F_j, F_j^d, X, $P_{11} = P_{11}^T$, $P_{22} = P_{22}^T$, P_{12}, $Q_{11} = Q_{11}^T$, $Q_{22} = Q_{22}^T$ and Q_{12}, and the scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\lambda_i > 0$ such that the following LMI-based conditions hold:

$$\Gamma_{ii} < 0, \quad \forall (i,j) \in \{1,...,r\}^2 \tag{15}$$

$$\frac{2}{r-1}\Gamma_{ii} + \Gamma_{ij} + \Gamma_{ji} < 0, \quad \forall (i,j) \in \{1,...,r\}^2 \; / i \neq j \tag{16}$$

$$\begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix} > 0 \tag{17}$$

$$\begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix} > 0 \tag{18}$$

with

$$\Gamma_{ij} = \begin{bmatrix} \tilde{\Theta} + \Upsilon_{ij} + \lambda_i \tilde{H}_i \tilde{H}_i^T & \tilde{E}_{ij}^T \\ \tilde{E}_{ij} & -\lambda_i I \end{bmatrix},$$

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} & P_{22} - Q_{12}^T & P_{11} + \bar{\tau}^2 Q_{12} \\ * & \tilde{\Theta}_{22} & -(1-\eta)P_{22} + Q_{12}^T & 0 \\ * & * & -Q_{11} & P_{12}^T \\ * & * & * & \bar{\tau}^2 Q_{22} \end{bmatrix},$$

$$\tilde{\Theta}_{11} = L + P_{12} + P_{12}^T + \bar{\tau}^2 Q_{11} - Q_{22},$$

$$\tilde{\Theta}_{12} = -(1-\eta)P_{12} + Q_{22},$$

$$\tilde{\Theta}_{22} = -(1-\eta)L - Q_{22},$$

$$\Upsilon_{ij} = \begin{bmatrix} \Upsilon_{ij}^{11} & \Upsilon_{ij}^{12} & 0 & -X + \varepsilon_2\left(X^T A_i^T + F_j^T B_i^T\right) \\ * & \Upsilon_{ij}^{22} & 0 & -\varepsilon_1 X + \varepsilon_2\left(X^T A_i^{dT} + F_j^{dT} B_i^T\right) \\ * & * & 0 & 0 \\ * & * & * & -\varepsilon_2(X + X^T) \end{bmatrix},$$

$$\Upsilon_{ij}^{11} = A_i X + X^T A_i^T + B_i F_j + F_j^T B_i^T,$$

$$\Upsilon_{ij}^{12} = A_i^d X + B_i F_j^d + \varepsilon_1\left(X^T A_i^T + F_j^T B_i^T\right),$$

$$\Upsilon_{ij}^{22} = \varepsilon_1\left(A_i^d X + X^T A_i^{dT} + B_i F_j^d + F_j^{dT} B_i^T\right),$$

$$\tilde{H}_i = \begin{bmatrix} H_i^T & \varepsilon_1 H_i^T & 0 & \varepsilon_2 H_i^T \end{bmatrix}^T,$$

$$\tilde{E}_{ij} = \begin{bmatrix} E_{ai}X + E_{bi}K_j & E_{ai}^d X + E_{bi}K_j^d & 0 & 0 \end{bmatrix}^T.$$

In that case, the changes of variables $K_j = F_j X^{-1}$ and $K_j^d = F_j^d X^{-1}$ provide the PDC control gains.

Proof: Let us consider the following LKF candidate:

$$V(t) = V_1(t) + V_2(t) + V_3(t) \tag{19}$$

with:

$$V_1(t) = \theta_1^T(t) M \theta_1(t), \tag{20}$$

$$V_2(t) = \int_{t-\tau(t)}^{t} x^T(s) S x(s) ds, \tag{21}$$

$$V_3(t) = \bar{\tau}\int_{-\bar{\tau}}^{0}\int_{t+s}^{t} \theta_2^T(w) N \theta_2(w) dw ds, \tag{22}$$

where $\theta_1(t) = \begin{bmatrix} x(t) \\ \int_{t-\tau(t)}^{t} x(s)ds \end{bmatrix}$ and $\theta_2(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$. The closed-loop system (10) is GAS if:

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) + \dot{V}_3(t) < 0 \tag{23}$$

Let us first focus on the time derivative of (20), one has:

$$\dot{V}_1(t) = \dot{\theta}_1^T(t) M \theta_1(t) + \theta_1^T(t) M \dot{\theta}_1(t) \tag{24}$$

Now, let $M = M^T = \begin{bmatrix} M_{11} & M_{12} \\ * & M_{22} \end{bmatrix} > 0$, (24) can be rewritten as:

$$\dot{V}_1(t) = 2\begin{bmatrix} x(t) \\ \int_{t-\tau(t)}^{t} x(s)ds \end{bmatrix}^T M \begin{bmatrix} \dot{x}(t) \\ \frac{d}{dt}\int_{t-\tau(t)}^{t} x(s)ds \end{bmatrix} \tag{25}$$

Since $\frac{d}{dt}\left(\int_{t-\tau(t)}^{t} x(s)ds\right) = x(t) - (1-\dot{\tau}(t))x(t-\tau(t))$, the equation (25) can be rewritten as:

$$\begin{aligned} \dot{V}_1(t) = {} & 2x^T(t) M_{11}\dot{x}(t) + 2\left(\int_{t-\tau(t)}^{t} x(s)ds\right)^T M_{12}^T \dot{x}(t) \\ & + 2x^T(t) M_{12} x(t) - 2(1-\dot{\tau}(t)) x^T(t) M_{12} x(t-\tau(t)) \\ & - 2(1-\dot{\tau}(t))\left(\int_{t-\tau(t)}^{t} x(s)ds\right)^T M_{22} x(t-\tau(t)) \\ & + 2\left(\int_{t-\tau(t)}^{t} x(s)ds\right)^T M_{22} x(t) \end{aligned} \tag{26}$$

Let us denote $\zeta(t) = \begin{bmatrix} x(t) \\ x(t-\tau(t)) \\ \int_{t-\tau(t)}^{t} x(s)ds \\ \dot{x}(t) \end{bmatrix}$, (26) yields:

$$\dot{V}_1(t) = \zeta^T(t) \Omega \zeta(t) \tag{27}$$

with

$$\Omega = \begin{bmatrix} M_{12} + M_{12}^T & \Omega^{12} & M_{22} & M_{11} \\ * & 0 & \Omega^{23} & 0 \\ * & * & 0 & M_{12}^T \\ * & * & * & 0 \end{bmatrix},$$

$$\Omega^{12} = -(1-\dot{\tau}(t)) M_{12},$$

$$\Omega^{23} = -(1-\dot{\tau}(t)) M_{22}.$$

Now, let us focus on the time derivative of (21), one has:

$$\begin{aligned} \dot{V}_2(t) &= x^T(t) S x(t) - (1-\dot{\tau}(t)) x^T(t-\tau(t)) S x(t-\tau(t)) \\ &= \zeta^T(t)\begin{bmatrix} S & 0 & 0 & 0 \\ 0 & -(1-\dot{\tau})S & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\zeta(t) \end{aligned} \tag{28}$$

Then, let us focus on the time derivative of (22), one has:

$$\dot{V}_3(t) = \bar{\tau}^2\theta_2^T(t)N\theta_2(t) - \bar{\tau}\int_{t-\bar{\tau}}^{t}\theta_2^T(s)N\theta_2(s)ds \quad (29)$$

Let $N = N^T = \begin{bmatrix} N_{11} & N_{12} \\ * & N_{22} \end{bmatrix} > 0$, then applying lemma 1 on the second right hand term of (29), one obtains:

$$\dot{V}_3(t) \leq \bar{\tau}^2\theta_2^T(t)N\theta_2(t) + \theta(t)^T\bar{N}\theta(t) \quad (30)$$

with $\bar{N} = \begin{bmatrix} -N_{22} & N_{22} & -N_{12}^T \\ N_{22} & -N_{22} & N_{12}^T \\ -N_{12} & N_{12} & -N_{11} \end{bmatrix}$ and with $\theta(t)$ defined in (11).
That is to say:

$$\dot{V}_3(t) \leq \zeta^T(t)\Psi\zeta(t) \quad (31)$$

with

$$\Psi = \begin{bmatrix} \bar{\tau}^2N_{11} - N_{22} & N_{22} & -N_{12}^T & \bar{\tau}^2N_{12} \\ * & -N_{22} & N_{12}^T & 0 \\ * & * & -N_{11} & 0 \\ * & * & * & \bar{\tau}^2N_{22} \end{bmatrix}$$

Thus, from (27), (28), (31) and since $\dot{\tau}(t) \leq \eta$, the condition (23) is satisfied if:

$$\zeta^T(t)\Theta\zeta(t) < 0 \quad (32)$$

with

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & M_{22} - N_{12}^T & M_{11} + \bar{\tau}^2N_{12} \\ * & \Theta_{22} & -(1-\eta)M_{22} + N_{12}^T & 0 \\ * & * & -N_{11} & M_{12}^T \\ * & * & * & \bar{\tau}^2N_{22} \end{bmatrix},$$

$$\Theta_{11} = S + M_{12} + M_{12}^T + \bar{\tau}^2N_{11} - N_{22},$$

$$\Theta_{12} = -(1-\eta)M_{12} + N_{22},$$

$$\Theta_{22} = -(1-\eta)S - N_{22}.$$

Next, let us rewrite (10) as:

$$\begin{array}{c} \mathcal{N}(t) = \left(\bar{A}_h + \bar{B}_hK_h\right)x(t) \\ + \left(\bar{A}_h^d + \bar{B}_hK_h^d\right)x\left(t-\tau(t)\right) - \dot{x}(t) = 0 \end{array} \quad (33)$$

From (33) one may introduce the slack variables R_1, R_2, and R_3, such that:

$$2\left(x^T(t)R_1 + x^T(t-\tau(t))R_2 + \dot{x}^T(t)R_3\right) \times \mathcal{N}(t) = 0 \quad (34)$$

which is equivalent to:

$$\zeta^T(t)\bar{\Upsilon}_{hh}\zeta(t) = 0 \quad (35)$$

with

$$\bar{\Upsilon}_{hh} = \begin{bmatrix} \bar{\Upsilon}_{hh}^{11} & \bar{\Upsilon}_{hh}^{12} & 0 & -R_1 + \left(\bar{A}_h + \bar{B}_hK_h\right)^T R_3^T \\ * & \bar{\Upsilon}_{hh}^{22} & 0 & -R_2 + \left(\bar{A}_h^d + \bar{B}_hK_h^d\right)^T R_3^T \\ * & * & 0 & 0 \\ * & * & * & -R_3 - R_3^T \end{bmatrix},$$

$$\bar{\Upsilon}_{hh}^{11} = R_1\left(\bar{A}_h + \bar{B}_hK_h\right) + \left(\bar{A}_h + \bar{B}_hK_h\right)^T R_1^T,$$

$$\bar{\Upsilon}_{hh}^{12} = R_1\left(\bar{A}_h^d + \bar{B}_hK_h^d\right) + \left(\bar{A}_h + \bar{B}_hK_h\right)^T R_2^T$$

$$\bar{\Upsilon}_{hh}^{22} = R_2\left(\bar{A}_h^d + \bar{B}_hK_h^d\right) + \left(\bar{A}_h^d + \bar{B}_hK_h^d\right)^T R_2^T$$

Thus, by summing (35) with (32), the condition (23) is satisfied if:

$$\Theta + \bar{\Upsilon}_{hh} < 0 \quad (36)$$

Let $X \in \mathbb{R}^{n\times n}$ be a free invertible matrice and $R_1 = X^{-T}$, $R_2 = \varepsilon_1X^{-T}$ and $R_3 = \varepsilon_2X^{-T}$ where ε_1 and ε_2 are two arbitrary scalars. Pre- and post-multiplying (36) respectively by $diag\begin{bmatrix} X & X & X & X \end{bmatrix}^T$ and its transpose, and with the change of variables $P_{ab} = X^TM_{ab}X$, $Q_{ab} = X^TN_{ab}X$ $\left(\forall (a,b) \in \{1,2\}^2, a \leq b\right)$, $L = X^TSX$, $F_h = K_hX$ and $F_h^d = K_h^dX$, then (36) becomes:

$$\tilde{\Theta} + \Upsilon_{hh} + \Delta\Upsilon_{hh} + \Delta\Upsilon_{hh}^T < 0 \quad (37)$$

with

$$\tilde{\Theta} = \begin{bmatrix} \tilde{\Theta}_{11} & \tilde{\Theta}_{12} & P_{22} - Q_{12}^T & P_{11} + \bar{\tau}^2Q_{12} \\ * & \tilde{\Theta}_{22} & -(1-\eta)P_{22} + Q_{12}^T & 0 \\ * & * & -Q_{11} & P_{12}^T \\ * & * & * & \bar{\tau}^2Q_{22} \end{bmatrix},$$

$$\tilde{\Theta}_{11} = L + P_{12} + P_{12}^T + \bar{\tau}^2Q_{11} - Q_{22},$$

$$\tilde{\Theta}_{12} = -(1-\eta)P_{12} + Q_{22},$$

$$\tilde{\Theta}_{22} = -(1-\eta)L - Q_{22},$$

$$\Upsilon_{hh} = \begin{bmatrix} \Upsilon_{hh}^{11} & \Upsilon_{hh}^{12} & 0 & -X + \varepsilon_2\left(X^TA_h^T + F_h^TB_h^T\right) \\ * & \Upsilon_{hh}^{22} & 0 & -\varepsilon_1X + \varepsilon_2\left(X^TA_h^{dT} + F_h^{dT}B_h^T\right) \\ * & * & 0 & 0 \\ * & * & * & -\varepsilon_2(X + X^T) \end{bmatrix},$$

$$\Upsilon_{hh}^{11} = A_hX + X^TA_h^T + B_hF_h + F_h^TB_h^T,$$

$$\Upsilon_{hh}^{12} = A_h^dX + B_hF_h^d + \varepsilon_1\left(X^TA_h^T + F_h^TB_h^T\right),$$

$$\Upsilon_{hh}^{22} = \varepsilon_1\left(A_h^dX + X^TA_h^{dT} + B_hF_h^d + F_h^{dT}B_h^T\right)$$

and

$$\Delta\Upsilon_{hh} = \begin{bmatrix} \Delta\Upsilon_{hh}^1 & \Delta\Upsilon_{hh}^2 & 0 & 0 \\ \varepsilon_1\Delta\Upsilon_{hh}^1 & \varepsilon_1\Delta\Upsilon_{hh}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \varepsilon_2\Delta\Upsilon_{hh}^1 & \varepsilon_2\Delta\Upsilon_{hh}^2 & 0 & 0 \end{bmatrix},$$

$$\Delta\Upsilon_{hh}^1 = \Delta A_hX + \Delta B_hF_h,$$

$$\Delta\Upsilon_{hh}^2 = \Delta A_h^dX + \Delta B_hF_h^d.$$

Expending $\Delta\Upsilon_{hh}$ with (2), (3) and (4), the inequatity (37) can be rewritten as:

$$\tilde{\Theta} + \Upsilon_{hh} + \tilde{H}_h\delta(t)\tilde{E}_{hh} + \tilde{E}_{hh}^T\delta^T(t)\tilde{H}_h^T < 0 \quad (38)$$

with $\tilde{H}_h = \begin{bmatrix} H_h^T & \varepsilon_1H_h^T & 0 & \varepsilon_2H_h^T \end{bmatrix}^T$ and $\tilde{E}_{hh} = \begin{bmatrix} E_{ah}X + E_{bh}K_h & E_{ah}^dX + E_{bh}K_h^d & 0 & 0 \end{bmatrix}^T$.
Then, applying lemma 3 and from (5), the inequality (38) is satisfied if:

$$\tilde{\Theta} + \Upsilon_{hh} + \lambda_h\tilde{H}_h\tilde{H}_h^T + \lambda_h^{-1}\tilde{E}_{hh}^T\tilde{E}_{hh} < 0 \quad (39)$$

where $\lambda_h > 0$ is a scalar function. Now, applying the Schur complement, yields:

$$\begin{bmatrix} \tilde{\Theta} + \Upsilon_{hh} + \lambda_h \tilde{H}_h \tilde{H}_h^T & \tilde{E}_{hh}^T \\ \tilde{E}_{hh} & -\lambda_h I \end{bmatrix} < 0 \tag{40}$$

Then, applying lemma 2, one obtains the conditions expressed in theorem 1. ■

Remark 2: The conditions expressed in theorem 1 are LMIs if the scalars ε_1 and ε_2 are prefixed. Note that, according to the proof of theorem 1 (see equation (34)) with the changes of variables ($R_2 = \varepsilon_1 X_1^{-T}$ and $R_3 = \varepsilon_2 X^{-T}$), the choice of these scalars is arbitrary. Nevertheless, they provide more degree of freedom to the design procedure. Hence in practice, to solve such kind of LMI conditions, these scalars are obtained from linear programming and searched in a logarithmically spaced family, e.g. $(\varepsilon_1, \varepsilon_2) \in \{10^{-6}, 10^{-5}, ..., 10^6\}^2$. As stated in [31], [32], this way of doing is generally outperforming the results obtained without these scalars.

Remark 3: One acknowledges that assumption 1, which is required to implement the control law (9), may be challenging in practical applications. Nevertheless a straightforward simplification of theorem 1 can be considered for constant delays by setting $\eta = 0$. In this case, if the constant time delay is known, the implementation of the controller (9) is no longer challenging. Moreover, if the time delay is available online but its variation rate is unknown, a straightforward simplification of theorem 1 hold with $L = 0$, $P_{12} = 0$ and $P_{22} = 0$.

IV. Illustrative example

In this section, our goal is to illustrate the effectiveness and the conservatism reduction of the result proposed in theorem 1 regarding to previous relevant results. Therefore, let us consider the following time varying delayed T-S system with 2 rules, defined by:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(z(t)) \left(\bar{A}_i x(t) + \bar{A}_i^d x(t - \tau(t)) + \bar{B}_i u(t) \right) \tag{41}$$

with

$$A_1 = \begin{bmatrix} 0 & 0.6 \\ 0 & a \end{bmatrix}, A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, A_1^d = \begin{bmatrix} 0.5 & 0.9 \\ 0 & 2 \end{bmatrix},$$

$$A_2^d = \begin{bmatrix} 0.9 & 0 \\ 1 & 1.6 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ b \end{bmatrix}, B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$H_1 = H_2 = \begin{bmatrix} -0.03 & 0 \\ 0 & 0.03 \end{bmatrix}$$

$$E_{a1} = E_{a2} = \begin{bmatrix} -0.15 & 0.2 \\ 0 & 0.04 \end{bmatrix}$$

$$E_{a1}^d = E_{a2}^d = \begin{bmatrix} -0.05 & -0.35 \\ 0.08 & -0.45 \end{bmatrix}$$

where a and b are two scalars dedicated to evaluate the feasibility fields, and the membership functions given by $h_1(x_1(t)) = 1/\left(1 + e^{-2(x_1(t)+\pi)}\right)$ and $h_2(x_1(t)) = 1 - h_2(x_1(t))$.

Figure 1 shows the feasibility fields of theorem 1 vs theorem 3.3 in [23] and theorem 3 in [22] for $\bar{\tau} = 0.2$, $\eta = 0.4$, $a \in [1, ..., 10]$ and $b \in [1, ..., 6]$. As one can notice, theorem 1 provides the widest feasibility field.

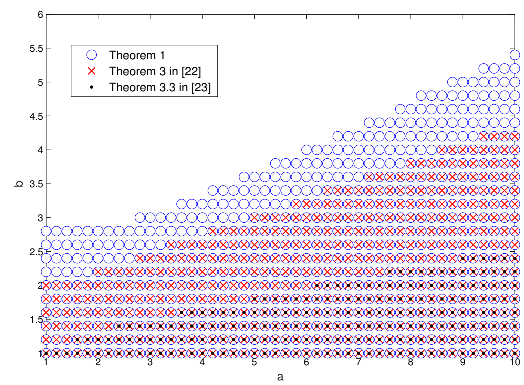

Fig. 1. Feasibility fileds obtained from theorem 1, theorem 3.3 in [23] and theorem 3 in [22].

Another way to compare theorem 1 with the existing results is to search the maximum allowable upper bound $\bar{\tau}$ of the time-delay $\tau(t)$. For the particular case where $a = 1$ and $b = 1$., table 1 shows the results obtained from theorem 1 and those proposed in [33], [26], [20], [22]. One more time, it is clear that theorem 1 provides the less conservative results.

TABLE I

Maximum allowable upper bound of $\tau(t)$

Considered results	$\eta = 0$	$\eta = 0.6$	$\eta = 3$	η unknown
Theorem 4 in [33]	-	-	-	0.3991
Theorem 3 in [26]	0.461	0.3941	0.2283	0.2321
Theorem 2 in [20]	-	0.6072	-	-
Theorem 3 in [22]	1.1644	1.0534	0.4780	0.4144
Theorem 1 in this paper	**1.1648**	**1.0594**	**2.1813**	**0.9632**

Hence, for $\bar{\tau} = 2.1813$ and $\eta = 3$, theorem 1 provides the following PDC gains for the control law (9):

$$K_1 = \begin{bmatrix} 2.157 & -19.633 \end{bmatrix}, K_2 = \begin{bmatrix} 1.156 & -18.572 \end{bmatrix},$$

$$K_1^d = \begin{bmatrix} 0.056 & -2.093 \end{bmatrix}, K_2^d = \begin{bmatrix} -1.022 & -1.746 \end{bmatrix}.$$

For simulation purpose, a random uncertain signal $\delta(t)$, plotted in figure 2, has been generated. Moreover in this figure, with a time varying delay $\tau(t) = \frac{\bar{\tau}}{2}\left(1 + sin\left(\frac{2\eta}{\bar{\tau}} t\right)\right)$, the initial state conditions $x_0 = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$ and $\forall t \in [-\bar{\tau}, 0], \phi(t) = x_0$, the state responses and the control signals are plotted from a closed-loop simulation. One notices that the closed-loop is stable despite the presence of uncertainties and time varying delay.

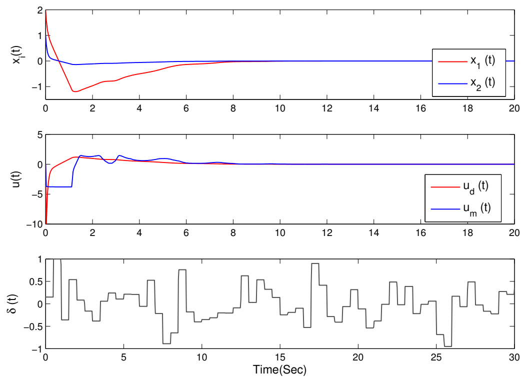

Fig. 2. Closed-loop simulation: state response $x_i(t)$, control signal $u(t)$ and uncertain signal $\delta(t)$

V. Conclusion

In this paper, new LMI conditions for the design of PDC controllers have been proposed for uncertain T-S fuzzy models with time-varying delay. The proposed PDC control law includes both memoryless and delayed state feedback. Based on a convenient Lyapunov-Krasoviskii functionals, delay-dependent conditions have been obtained. A numerical example has been provided to illustrate the effectiveness of the results, as well as its improvement in terms of conservatism regarding to recent studies.

References

[1] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Transactions on Systems, Man and Cybernetics*, no. 1, pp. 116–132, 1985.

[2] K. Tanaka and H. O. Wang, *Fuzzy control systems design and analysis: a linear matrix inequality approach*. John Wiley & Sons, 2001.

[3] P. Gahinet, P. Apkarian, and M. Chilali, "Affine parameter-dependent lyapunov functions and real parametric uncertainty," *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 436–442, 1996.

[4] T. Bouarar, K. Guelton, B. Mansouri, and N. Manamanni, "Lmi stability conditions for takagi-sugeno uncertain descriptors," in *IEEE International Fuzzy Systems Conference*, 2007.

[5] M. Zerar, K. Guelton, and N. Manamanni, "Linear fractional transformation based h-infinity output stabilization for takagi-sugeno fuzzy models," *Mediterranean Journal of Measurement and Control*, vol. 4, no. 3, pp. 111–121, 2008.

[6] T. Bouarar, K. Guelton, and N. Manamanni, "Robust non-quadratic static output feedback controller design for takagi–sugeno systems using descriptor redundancy," *Engineering Applications of Artificial Intelligence*, vol. 26, no. 2, pp. 739–756, 2013.

[7] D. Jabri, K. Guelton, N. Manamanni, A. Jaadari, and C.-D. Chinh, "Robust stabilization of nonlinear systems based on a switched fuzzy control law," *Journal of Control Engineering and Applied Informatics*, vol. 14, no. 2, pp. 40–49, 2012.

[8] H. D. Tuan, P. Apkarian, T. Narikiyo, and Y. Yamamoto, "Parameterized linear matrix inequality techniques in fuzzy control system design," *IEEE Trans. on fuzzy systems*, vol. 9, no. 2, pp. 324–332, 2001.

[9] A. Sala, "On the conservativeness of fuzzy and fuzzy-polynomial control of nonlinear systems," *Ann. Rev. in Control*, vol. 33, no. 1, pp. 48–58, 2009.

[10] C.-C. Duong, K. Guelton, and N. Manamanni, "A sos based alternative to lmi approaches for non-quadratic stabilization of continuous-time takagi-sugeno fuzzy systems," in *IEEE International Conference on Fuzzy Systems*, 2013, pp. 1–7.

[11] J.-P. Richard, "Time-delay systems: an overview of some recent advances and open problems," *automatica*, vol. 39, no. 10, pp. 1667–1694, 2003.

[12] Y.-Y. Cao and P. M. Frank, "Analysis and synthesis of nonlinear time-delay systems via fuzzy control approach," *IEEE Transactions on Fuzzy Systems*, vol. 8, no. 2, pp. 200–211, 2000.

[13] A. Chen and J. Wang, "Delay-dependent l2–linf control of linear systems with multiple time-varying state and input delays," *Nonlinear Analysis: Real World Applications*, vol. 13, no. 1, pp. 486–496, 2012.

[14] S.-H. Tsai, Y.-A. Chen, and J.-C. Lo, "A novel stabilization condition for a class of t–s fuzzy time-delay systems," *Neurocomputing*, vol. 175, pp. 223–232, 2016.

[15] Z. Zhang, C. Lin, and B. Chen, "New stability and stabilization conditions for t–s fuzzy systems with time delay," *Fuzzy Sets and Systems*, vol. 263, pp. 82–91, 2015.

[16] L. Zhao, H. Gao, and H. R. Karimi, "Robust stability and stabilization of uncertain t–s fuzzy systems with time-varying delay: an input–output approach," in *IEEE International Conference on Fuzzy Systems*, vol. 21, no. 5, 2013, pp. 883–897.

[17] H.-B. Zeng, J. H. Park, J.-W. Xia, and S.-P. Xiao, "Improved delay-dependent stability criteria for t–s fuzzy systems with time-varying delay," *Applied Math. and Computation*, vol. 235, pp. 492–501, 2014.

[18] P.-L. Liu, "A delay decomposition approach to robust stability analysis of uncertain systems with time-varying delay," *ISA transactions*, vol. 51, no. 6, pp. 694–701, 2012.

[19] C. Peng and Y.-C. Tian, "Delay-dependent robust stability criteria for uncertain systems with interval time-varying delay," *Journal of Computational and Applied Mathematics*, vol. 214, no. 2, pp. 480–494, 2008.

[20] C. Peng, Y.-C. Tian, and E. Tian, "Improved delay-dependent robust stabilization conditions of uncertain t–s fuzzy systems with time-varying delay," *Fuzzy sets and systems*, vol. 159, no. 20, pp. 2713–2729, 2008.

[21] R.-J. Wang, W.-W. Lin, and W.-J. Wang, "Stabilizability of linear quadratic state feedback for uncertain fuzzy time-delay systems," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 34, no. 2, pp. 1288–1292, 2004.

[22] L. Li and X. Liu, "New approach on robust stability for uncertain t–s fuzzy systems with state and input delays," *Chaos, Solitons & Fractals*, vol. 40, no. 5, pp. 2329–2339, 2009.

[23] J. Yoneyama, "New delay-dependent approach to robust stability and stabilization for takagi–sugeno fuzzy time-delay systems," *Fuzzy Sets and Systems*, vol. 158, no. 20, pp. 2225–2237, 2007.

[24] C. Peng and M.-R. Fei, "An improved result on the stability of uncertain t-s fuzzy systems with interval time-varying delay," *Fuzzy Sets and Systems*, vol. 212, pp. 97–109, 2013.

[25] F. Bourahala, K. Guelton, F. Khaber, and N. Manamanni, "Improvements on pdc controller design for takagi-sugeno fuzzy systems with state time-varying delays," in *4th IFAC International Conference on Intelligent Control and Automation Sciences*, Reims, France, 2016.

[26] H. Gassara, A. El Hajjaji, and M. Chaabane, "Robust control of t-s fuzzy systems with time-varying delay using new approach," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 14, pp. 1566–1578, 2010.

[27] Y. S. Moon, P. Park, W. H. Kwon, and Y. S. Lee, "Delay-dependent robust stabilization of uncertain state-delayed systems," *International Journal of control*, vol. 74, no. 14, pp. 1447–1455, 2001.

[28] K. Gu, J. Chen, and V. L. Kharitonov, *Stability of time-delay systems*. Springer Science & Business Media, 2003.

[29] M. Wu, Y. He, and J.-H. She, *Stability analysis and robust control of time-delay systems*. Springer, 2010.

[30] L. Xie, "Output feedback hinf control of systems with parameter uncertainty," *International Journal of control*, vol. 63, no. 4, pp. 741–750, 1996.

[31] R. C. Oliveira, M. C. De Oliveira, and P. L. Peres, "Robust state feedback lmi methods for continuous-time linear systems: Discussions, extensions and numerical comparisons," in *IEEE Int. Symp. on Computer-Aided Control System Design*, 2011.

[32] A. Jaadari, T.-M. Guerra, A. Sala, M. Bernal, and K. Guelton, "New controllers and new designs for continuous-time takagi-sugeno models." in *IEEE International Conference on Fuzzy Systems*, 2012, pp. 1–7.

[33] Y. Zhao, H. Gao, J. Lam, and B. Du, "Stability and stabilization of delayed t–s fuzzy systems: A delay partitioning approach," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 4, pp. 750–762, 2009.