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# A SOS Based Alternative To LMI Approaches For Non-Quadratic Stabilization Of Continuous-Time Takagi-Sugeno Fuzzy Systems

Chinh-Cuong Duong, Kevin Guelton, Nouredine Manamanni  
Université de Reims Champagne-Ardenne  
CReSTIC EA3804, Moulin de la Housse BP1034  
51687 Reims, France  
kevin.guelton@univ-reims.fr

**Abstract**—Nowadays, when dealing with non-quadratic controllers design for continuous-time Takagi-Sugeno (TS) models, LMIs-based successive conditions become more and more complex for a conservatism reduction that is sometime questionable. Therefore, in this paper it is assumed that it should be interesting to explore what can be done, else than LMIs, in the non-quadratic framework. Indeed, in most of the cases, non-quadratic LMIs suffer from the requirement of unknown parameters or lead to local stability analysis. Hence, the aim of this paper is to show, at a first attempt, that the Sum-Of-Squares formalism is suitable to design non-PDC controllers which stabilizing TS models on their whole definition set. However, it is pointed-out that the SOS formalism requires a restrictive modeling assumption, understood as a drawback but opening some possible further prospects.

**Keywords**—Takagi-Sugeno fuzzy models; Non-quadratic stabilization; Non-PDC controller design; Sum-Of-Squares.

## I. INTRODUCTION

Takagi-Sugeno (TS) models have been the subject of many researches in the past few decades. Indeed, firstly introduced in 1985 [1], they were originally based on an IF-THEN rules fuzzy formalism to approximate nonlinear systems. Then, by the use of the Sector Nonlinearity modeling approach [2], the interest on TS models has grown up due to their ability to match a nonlinear system exactly on a compact set of the state space. Indeed, similarly to Quasi-LPV (Linear Parameter Varying) models, a TS model remains on a convex polytopic representation of a nonlinear system, i.e. a collection of linear systems blended together by convex nonlinear functions. Based on the convex property, numerous works have been done to study TS models stability or stabilization through the optimization of Linear Matrix Inequality (LMI) conditions obtained from the direct Lyapunov methodology, see e.g. [3-6]

Most of LMI based studies on TS models stability analysis, controller or observer design, were obtained through the use of a common quadratic Lyapunov function. However, these works suffer from conservatism since a common Lyapunov matrix has to be solution of a set of several LMI constraints.

For a review of conservatism sources in TS studies, one may refer to [7]. In order to reduce the conservatism, some LMI relaxation schemes have been proposed [8-9] and recent studies have focused on the use of Non-Quadratic Lyapunov Functions (NQLF), see e.g. [10-11]. These NQLF, also called multiple Lyapunov or fuzzy Lyapunov functions, appear convenient since they are based on similar convex polytopic structures as the TS models to be analyzed or stabilized. However, when dealing with continuous-time TS fuzzy models, such non-quadratic LMI approaches remain difficult to apply since they often require some parameters (bounds of the derivatives of the membership functions) which are often unknown in practice. To overcome this drawback, line integral Lyapunov functions has been employed leading to LMIs with a particular structure of Lyapunov decision variables (off-diagonal common matrices) which may be understood as a source of conservatism [12-14]. Some other recent works reduced to local non-quadratic controller design for continuous-time TS models have been proposed, see e.g. [15]. However they lead to a somewhat complex LMI formulation and the global non-quadratic stabilization remains an open problem.

Complementary to LMI based conditions for TS models, some recent works were focused on the stability analysis and the stabilization of polynomial fuzzy models using Sum-Of-Squares (SOS) techniques [16-17]. Following the way of LMI studies regarding to conservatism reduction using non-quadratic Lyapunov functions, these results have been extended to the stability analysis via multiple polynomial Lyapunov functions [18-20].

Nowadays, it is sometime argued that LMI based successive improvements lead to more and more complex problem formulations for conservatism reductions that are sometime minor and so, it may be interesting to investigate other ways to formulate the non-quadratic stabilization problem of TS models. In this context, one doesn't pretend to overcome existing non-quadratic LMI approaches in terms of conservatism but just exploring another ways to partially answer the question "what else than LMIs in non-quadratic stabilization of TS fuzzy models?". Therefore, as a first attempt, the aim of this paper is to show how SOS based

approach can be used to derive global non-quadratic controller design conditions for conventional TS models, which can be seen as a special case of polynomial ones, as an alternative to LMIs without requiring unknown parameters, even if another restrictive assumption is still required in stabilization (see assumption 1 below) [16-17].

The rest of the paper is organized as follows: First, usual non-quadratic approaches are recalled and discussed. Then, SOS based conditions for continuous-time TS models using non-quadratic Lyapunov functions are provided. Finally, a numerical example illustrates the validity of the proposed approach.

## II. LMI BASED NON-QUADRATIC STABILIZATION AND PROBLEM STATEMENT

Consider the following nonlinear system:

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t) \quad (1)$$

where  $x(t) \in \mathfrak{R}^n$  is the state vector,  $u(t) \in \mathfrak{R}^m$  is the input vector,  $A(x(t)) \in \mathfrak{R}^{n \times n}$  and  $B(x(t)) \in \mathfrak{R}^{n \times m}$  are matrices which entries may contains smooth and bounded nonlinear functions on a compact set  $\Omega \subseteq \mathfrak{R}^n$  of the state space.

Using the well-known sector nonlinearity approach [2], (1) can be exactly rewritten on  $\Omega$  as a TS fuzzy system such that:

$$\dot{x}(t) = \sum_{i=1}^r h_i(z(t)) (A_i x(t) + B_i u(t)) \quad (2)$$

where  $z(t) \in \mathfrak{R}^p$  is the premise vector,  $A_i \in \mathfrak{R}^{n \times n}$  and  $B_i \in \mathfrak{R}^{n \times m}$  are constant matrices and  $h_i(z(t)) > 0$  are positive fuzzy membership functions holding the convex sum property  $\sum_{i=1}^r h_i(z(t)) = 1$ .

Now, let us consider the following non-PDC control law [10]:

$$u(t) = \sum_{i=1}^r h_i(z(t)) F_i \left( \sum_{i=1}^r h_i(z(t)) X_i \right)^{-1} x(t) \quad (3)$$

where  $F_i \in \mathfrak{R}^{m \times n}$  and  $X_i \in \mathfrak{R}^{n \times n}$  are gain matrices to be synthesized.

**Remark 1:** From the sector nonlinearity approach [2], the TS fuzzy model (2) is reputed to be valid on a local subset  $\Omega$  of the state space if some nonlinearities of (1) are unbounded on

the whole state space. Nevertheless, if all the nonlinearities are bounded on the whole state space, (2) is said to be globally matching (1). Therefore, in the sequel as well as for all TS fuzzy model based controller design studies, a non-PDC controller (3) is assumed to be globally stabilizing the TS fuzzy system (2) on its whole definition set  $\Omega$  instead of the whole state space of the initial nonlinear model (1).

**Notations:** In the sequel, to lighten mathematical expressions, when there is no ambiguity, the time  $t$  will be omitted. Moreover, for a matrix  $M$ , one denotes  $He(M) = M + M^T$ ,

$M_h = \sum_{i=1}^r h_i(z(t)) M_i$ ,  $M_{hh} = \sum_{i=1}^r \sum_{j=1}^r h_i(z(t)) h_j(z(t)) M_{ij}$  and so on.

From (2) and (3), the closed loop dynamics may be expressed as:

$$\dot{x} = \sum_{i=1}^r \sum_{j=1}^r h_i(z) h_j(z) \left( A_i + B_i F_j \left( \sum_{i=1}^r h_i(z) X_i \right)^{-1} \right) x \quad (4)$$

In order to design a non-PDC controller (3) such that the closed-loop dynamics (4) is stable, the direct Lyapunov method has been used in several studies to lead to LMI conditions, see e.g.[10]. Among Lyapunov function candidates, the NQLF is the subject of many interest since its structure shares the same fuzzy membership functions as the TS model to be stabilized and so leads less conservative LMI conditions than common quadratic Lyapunov functions [3]. Therefore, for the stabilization of (4), one considers the NQLF given by [10]:

$$v(x) = x^T \left( \sum_{i=1}^r h_i(z) X_i \right)^{-1} x \quad (5)$$

which is strictly positive for  $x(t) \neq 0$  with  $X_i = X_i^T > 0$ .

The following double sums relaxation lemma will be used in further proofs to reduce the conservatism of Lyapunov based controller design conditions.

**Lemma 1** [9]: The inequality:

$$\sum_{i=1}^r \sum_{j=1}^r h_i(z) h_j(z) \Upsilon_{ij} < 0 \quad (6)$$

holds if,  $\forall (i, j, k) \in \{1, \dots, r\}^3 / j \neq i$ , the inequalities (7) and (8) are verified.

$$\Upsilon_{ii} < 0, \quad (7)$$

$$\frac{1}{r-1} \Upsilon_{ii} + \frac{1}{2} (\Upsilon_{ij} + \Upsilon_{ji}) < 0 \quad (8)$$

The following theorem, inspired from [11], expresses non-quadratic LMI conditions for the design of non-PDC controllers (3).

**Theorem 1:** Assuming  $\forall i \in \{1, \dots, r\}, |\dot{h}_i(z)| < \theta_i$ , the TS fuzzy system (2) is asymptotically stabilized by the non-PDC control law (3) if there exist the matrices  $F_j$ ,  $X_i = X_i^T > 0$  and  $R_{ij}$  such that,  $\forall (i, j, k) \in \{1, \dots, r\}^3 / j \neq i$ , the LMIs (7), (8) and  $X_k + R_{ij} > 0$  are satisfied with :

$$\Upsilon_{ij} = He(A_i X_j + B_i F_j) - \sum_{k=1}^r \theta_k (X_k + R_{ij}) \quad (9)$$

**Proof:** From (5), the closed-loop dynamics (4) is stable if  $\dot{v}(x) < 0$ , i.e.:

$$x^T \left( He(X_h^{-1} A_h + X_h^{-1} B_h F_h X_h^{-1}) + \dot{X}_h^{-1} \right) x < 0 \quad (10)$$

That is to say,  $\forall x$  :

$$He(X_h^{-1} A_h + X_h^{-1} B_h F_h X_h^{-1}) + \dot{X}_h^{-1} < 0 \quad (11)$$

Multiplying (11) left and right by  $X_h$ , one obtains:

$$He(A_h X_h + B_h F_h) + X_h \dot{X}_h^{-1} X_h < 0 \quad (12)$$

Now, since  $\dot{X}_h = -X_h \dot{X}_h^{-1} X_h$ , see e.g. [22], the inequality (12) can be rewritten as:

$$He(A_h X_h + B_h F_h) - \dot{X}_h < 0 \quad (13)$$

Note that the term  $\dot{X}_h$  occurs in (13) and so a bounded real lemma cannot be obtained directly in terms of LMIs. To cope with this problem, a way consists on considering the bounds of the membership functions' derivatives  $|\dot{h}_i(z)| < \theta_i$ . Indeed:

$$\dot{X}_h = \sum_{i=1}^r \dot{h}_i(z) X_i \quad (14)$$

Moreover, following recent non-quadratic improvement regarding to conservatism [11], slack decision matrices may be introduced since, whatever a matrix  $R_{hh}$  is:

$$\sum_{i=1}^r \dot{h}_i(z) = \sum_{k=1}^r \dot{h}_k(z) \left( \sum_{i=1}^r \sum_{j=1}^r h_i(z) h_j(z) R_{ij} \right) = 0 \quad (15)$$

and so:

$$\dot{X}_h = \sum_{i=1}^r \sum_{j=1}^r \sum_{k=1}^r h_i(z) h_j(z) \dot{h}_k(z) (X_k + R_{ij}) \quad (16)$$

Therefore, assuming the bounds of membership functions derivatives  $|\dot{h}_i(z)| < \theta_i$  and from (16), the inequality (13) is satisfied if:

$$\Upsilon_{hh} = \sum_{i=1}^r \sum_{j=1}^r h_i(z) h_j(z) \Upsilon_{ij} < 0 \quad (17)$$

with  $\Upsilon_{ij} = He(A_i X_j + B_i F_j) - \sum_{k=1}^r \theta_k (X_k + R_{ij})$  and, for all  $(i, j, k) \in \{1, \dots, r\}^3$ ,  $X_k + R_{ij} > 0$ .

Thus, from (17) and the relaxation lemma 1, one can express a bounded real lemma expressed in theorem 1.  $\square$

**Remark 2:** The bounded real lemma given in theorem 1 constitutes a slight improvement of the LMI conditions proposed in [11]. Indeed, the latter study's LMI conditions may be recovered from theorem 1 by considering the slack variables  $R_{ij} = R$  common for each LMIs. Moreover, if  $R_{ij} = 0$ , one obtains the non-quadratic conditions proposed in [10]. Then, considering  $X_k = X$  and  $R_{ij} = -X$  common matrices, it yields the well-known quadratic conditions [3][9]. Consequently, theorem 1 includes as special cases some previous results and obviously leads to less conservatism.

**Remark 3:** Theorem 1 requires the knowledge of the bounds  $\theta_i$  of the time derivatives of the membership functions  $\dot{h}_i(z(t))$  as parameters to the LMI computation. It is often criticized since the membership functions depend on the time evolution premises variables, which are often state variables. Therefore, in practice, it is sometime difficult, even more for stabilization, to estimate these bounds before solving the LMIs and so before having synthesized the closed-loop dynamics.

Moreover, without prior knowledge on the closed-loop state dynamics, it is not correct to tell about global asymptotical stability on  $\Omega$  since it is somewhat hypothetical to say that  $|\dot{h}_i(z)| < \theta_i$  arises for every initial condition  $x(0) \in \Omega$ .

**Remark 4:** Other ways to cope with the problem of the membership functions' derivatives depicted in remark 2 have been proposed. First, a Line-integral fuzzy Lyapunov functional (LIFLF) has been proposed in [12-14]. However, even if some recent improvement lead to LMIs in stabilization [14], these approaches need to recast the decision variables to a particular case where the off-diagonal elements of all  $X_i$  are common and thus leading to conservatism. More recently, a local result has been proposed to overcome the problem of the membership functions' derivatives [15]. In the latter study, the idea is to maximize a domain of attraction

$D = \left\{ x : x \in \Omega, \left\| \sum_{j=1}^r \dot{h}_j P_j \right\| < \lambda \right\}$ . Although this result is LMI (with a somewhat complex formulation), it is still a local asymptotical stabilization approach since here, the domain of attraction  $D$  doesn't recover the whole set  $\Omega$ .

At this step, the concerns pointed out in remarks 3 and 4 are understood as drawbacks of LMI based non-quadratic controller design for TS models. Moreover, since LMI based successive improvements leads to more and more complex problem formulations for conservatism reductions that are sometime minor, it may be interesting to investigate other ways to formulate the non-quadratic stabilization problem of TS models. This is the attempt of the following section, which constitutes the main contribution of the paper, where one proposes Sum-Of-Squares (SOS) based global conditions (on  $\Omega$ ) as a possible alternative to LMIs.

### III. SOS BASED NON-QUADRATIC CONTROLLER DESIGN FOR TS MODELS

In this section, the goal is to propose new non-quadratic conditions for the design of non-PDC controllers (3) for TS Fuzzy systems (2) leading to global stabilization (on  $\Omega$ ) of the closed-loop dynamics (4). These conditions will be based on the SOS formalism [23] with the NQLF candidate (5).

For more clarity of further mathematical proofs, before presenting the main results, some useful preliminaries (Assumptions, Definitions and Lemmas) are presented.

**Assumption 1:** Following previous work on polynomial fuzzy models stabilization [16-17], to avoid non-convex conditions, we assume that the premise vector  $z$  only depends on the states that are not directly affected by the control input, namely, states whose corresponding rows in  $B_i$  are zero. Let

$A_i^k$  be the  $k^{\text{th}}$  row of  $A_i$ .  $K = \{k_1, k_2, \dots, k_p\}$  denotes the set of row indices of  $B_i$  whose corresponding rows are equal to zero, one can write:

$$\dot{x}_k = \sum_{i=1}^r h_i(z) A_i^k x, \forall k \in K \quad (18)$$

and  $\forall s \in \{1, \dots, r\}$ ,

$$\frac{\partial h_i(z)}{\partial x_s} = 0, \forall s \in \{1, \dots, n\} / K \quad (19)$$

**Assumption 2:**  $\forall i \in \{1, \dots, r\}$ ,  $h_i(z)$  are continuously derivable within each states variables  $x_s$  (with  $s \in K$ ) on the compact set  $\Omega$ . Thus, their derivatives  $g_i^s(z) = \frac{\partial h_i(z)}{\partial x_s}$  are bounded on  $\Omega$  such that  $\forall t, g_k^s(z) \in [\alpha_{k1}^s, \alpha_{k2}^s]$  and these bounds can always be known on  $\Omega$  whereas the bounds  $|\dot{h}_i(z)| < \theta_i$  are unknown. Therefore, one may applies the sector nonlinearity approach and one can define convex functions  $\omega_{k1}^s(z) \geq 0$  and  $\omega_{k2}^s(z) \geq 0$ ,  $\omega_{k1}^s + \omega_{k2}^s = 1$ , such that:

$$g_k^s(z) = \frac{\partial h_k(z)}{\partial x_s} = \sum_{\tau=1}^2 \omega_{k\tau}^s(z) \alpha_{k\tau}^s \quad (20)$$

**Example:** To illustrate assumption 2, let us consider the membership function  $h_1(x_1) = \sin^2 x_1$ . Its time derivative  $\dot{h}_1(x_1) = 2\dot{x}_1 \cos x_1 \sin x_1$  depends on  $\dot{x}_1$  and so its bound  $|\dot{h}_1(z)| < \theta_1$  is unknown for stabilization studies since the dynamics  $\dot{x}_1$  is unknown before designing the non-PDC controller and so the closed-loop dynamics. However,  $g_1^1(x_1) = \frac{\partial h_1(x_1)}{\partial x_1} = 2 \cos x_1 \sin x_1 \in [\alpha_{11}^1, \alpha_{12}^1]$  is bounded whatever  $x_1$  is, and  $\alpha_{11}^1 = -1$  and  $\alpha_{12}^1 = 1$  are known parameters! Hence, considering  $\omega_{11}^1(x_1) = \frac{1}{2} - \cos x_1 \sin x_1 \geq 0$

and  $\omega_{12}^1(x_1) = \frac{1}{2} + \cos x_1 \sin x_1 \geq 0$ , such that  $\omega_{11}^1 + \omega_{12}^1 = 1$ ,

one has  $g_1^1(x_1) = \frac{\partial h_1(x_1)}{\partial x_1} = \sum_{\tau=1}^2 \omega_{1\tau}^1(z) \alpha_{1\tau}^1$ .

The following theorem expresses the non-quadratic SOS based conditions for the design of non-PDC controllers (3) which globally stabilize (2) on  $\Omega$  under assumption 1.

**Theorem 2:** The TS fuzzy system (2) is globally asymptotically stabilized (on  $\Omega$ ) by the non-PDC control law (3) if there exist the matrices  $F_j$ ,  $X_i = X_i^T > 0$  and  $R_j$  as well as the polynomials  $\varepsilon_{1i}(x) > 0$ ,  $\varepsilon_{2i}(x) \geq 0$ ,  $\varepsilon_{3ij}(x) \geq 0$  and  $\varepsilon_{4ij}(x) \geq 0$  such that,  $\forall (i, j, k) \in \{1, \dots, r\}^3$ ,  $j \neq i$ ,  $\forall s \in K$  and  $\forall \tau \in \{1, 2\}$ , the SOS conditions (21), (22), (23) and (24) are satisfied.

$$x^T (X_i - \varepsilon_{1i}) x \text{ is SOS} \quad (21)$$

$$x^T (X_k + R_j - \varepsilon_{2kj}) x \text{ is SOS} \quad (22)$$

$$-x^T [\mathbf{P}_{iips}^q + \varepsilon_{2i} I] x \text{ is SOS} \quad (23)$$

$$-x^T \left( \frac{1}{r-1} \mathbf{P}_{iips}^q + \frac{1}{2} (\mathbf{P}_{iips}^q + \mathbf{P}_{jips}^q) + \varepsilon_{3ij} I \right) x \text{ is SOS} \quad (24)$$

with  $\mathbf{P}_{ijk\tau}^s = He(A_i X_j + B_i F_j) - \alpha_{k\tau}^s A_i^s x (X_k + R_j)$ .

**Proof:** Let us consider the NQLF candidate (3). The TS system (2) is stable if (13) holds, i.e.  $\forall x$ :

$$x^T (He(A_h X_h + B_h F_h) - \dot{X}_h) x < 0 \quad (25)$$

Let us now focus on the term  $\dot{X}_h$  occurring in (25). Similarly to equation (16), one can write:

$$\begin{aligned} \dot{X}_h &= \sum_{j=1}^r \sum_{k=1}^r h_j(z) \dot{h}_k(z) (X_k + R_j) \\ &= \sum_{j=1}^r \sum_{k=1}^r \sum_{s \in K} h_j(z) \frac{\partial h_k(z)}{\partial x_s} \dot{x}_s (X_k + R_j) \end{aligned} \quad (26)$$

Under assumptions 2,  $\forall t, g_k^s(z) \in [\alpha_{k1}^s, \alpha_{k2}^s]$  with known bounds on the compact set  $\Omega$ . Therefore, applying the sector nonlinearity approach on  $g_k^s(z)$ , there always exist convex functions  $\omega_{k1}^s(z) > 0$  and  $\omega_{k2}^s(z) > 0$ ,  $\omega_{k1}^s + \omega_{k2}^s = 1$ , such that (20) holds. Thus, equation (26) can be rewritten as:

$$\dot{X}_h = \sum_{j=1}^r \sum_{k=1}^r \sum_{s \in K} \sum_{\tau=1}^2 h_j(z) \omega_{k\tau}^s(z) \alpha_{k\tau}^s \dot{x}_s (X_k + R_j) \quad (27)$$

Now, under assumption 1 and substituting (18) in (27), it yields:

$$\dot{X}_h = \sum_{i=1}^r \sum_{j=1}^r \sum_{k=1}^r \sum_{s \in K} \sum_{\tau=1}^2 h_i(z) h_j(z) \omega_{k\tau}^s(z) \alpha_{k\tau}^s A_i^s x (X_k + R_j) \quad (28)$$

Thus, under assumption 1 and 2, considering (28), the inequality (25) is, strictly equivalent on  $\Omega$  to:

$$x^T \sum_{i=1}^r \sum_{j=1}^r \sum_{k=1}^r \sum_{s \in K} \sum_{\tau=1}^2 h_i(z) h_j(z) \omega_{k\tau}^s(z) \mathbf{P}_{ijk\tau}^s x < 0 \quad (29)$$

with  $\mathbf{P}_{ijk\tau}^s = He(A_i X_j + B_i F_j) - \alpha_{k\tau}^s A_i^s x (X_k + R_j)$

Note that  $x^T \mathbf{P}_{ijk\tau}^s x$  are scalar polynomials in  $x$ , so the negativity of (29), and so (25), may be checked using SOS optimization tools. Therefore, applying lemma 1 on (29), one obtains the SOS conditions summarized in theorem 2.  $\square$

**Remark 4:** The most improvements of the above-defined SOS conditions (theorem 2) regarding to LMI ones (e.g. theorem 1) are:

- The conditions of theorem 2 are free of unknown parameters such as the bounds of derivatives of the membership functions. Indeed, the bounds of  $g_k^s(z)$ ,  $\alpha_{k1}^s$  and  $\alpha_{k2}^s$  do not depend directly on  $\dot{x}$  and are always known and well-defined under assumption 2 (see e.g. the example provided above).
- The whole parameters of theorem 2 being known, the SOS based approach ensures, when a solution exists, a global stabilization on the whole set  $\Omega$ , where the TS system is valid, which cannot be guarantee with LMI based approaches such that [10-11] or [15] (since they are local results), excepted LMI conditions obtained through a line-integral Lyapunov function [14]. However, in the latter approach, the particular case with common off-diagonal decision matrices is required.

However, it appears that the main drawback of SOS based approaches relies on assumption 1, namely the premise vector  $z$  only depends on the states that are not directly affected by

the control input. Obviously, it is restrictive and, as state in [16] regarding to the stabilization of polynomial fuzzy models, it may be advantageous to employ an initial state transformation to introduce as many zero rows as possible in the input matrices  $B_i$ . This point, understood as a lock, should be the focus of further SOS based developpement dealing with controller design for fuzzy models.

#### IV. NUMERICAL EXAMPLE

Let us consider the following TS system:

$$\dot{x}(t) = \sum_{i=1}^2 h_i(z) (A_i x(t) + B_i u(t)) \quad (30)$$

where  $A_1 = \begin{bmatrix} 2 & -10 \\ 2 & 0 \end{bmatrix}$ ,  $A_2 = \begin{bmatrix} 1 & -5 \\ 1 & 2 \end{bmatrix}$ ,  $B_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ ,  
 $B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ ,  $h_1(x_1) = \frac{1}{2}(1 - \sin x_1)$  and  
 $h_2(x_1) = 1 - h_1(x_1)$ .

Recall that, using theorem 1 and previous LMI-based non-quadratic conditions, a non-PDC controller guaranteeing a global stabilization cannot be designed, excepted LMI conditions obtained through a line-integral Lyapunov function [14]. Moreover, as illustrated by the phase portrait plot in figure 1, the open-loop TS system (30) is unstable.

From the definition of  $h_1$  and  $h_2$ , one has :

$$\frac{\partial h_1(x_1)}{\partial x_1} = -\frac{1}{2} \cos x_1 \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \quad (31)$$

$$\frac{\partial h_2(x_1)}{\partial x_1} = \frac{1}{2} \cos x_1 \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \quad (32)$$

Therefore, one may apply theorem 2 with the known parameters  $\alpha_{11}^1 = \alpha_{21}^1 = -\frac{1}{2}$  and  $\alpha_{12}^1 = \alpha_{22}^1 = \frac{1}{2}$ . Note that, the decision variables  $\varepsilon_{1i}$ ,  $\varepsilon_{2i}$ ,  $\varepsilon_{3ij}$  and  $\varepsilon_{4ij}$  have been set as zero-order polynomials. The result, obtained using the toolbox SOSTOOLS for Matlab [23], is given by the matrices:

$$F_1 = \begin{bmatrix} -20.608 & -2.9356 \end{bmatrix}, F_2 = \begin{bmatrix} -20.8868 & -1.0598 \end{bmatrix},$$

$$X_1 = \begin{bmatrix} 7.9772 & -2.068 \\ -2.068 & 0.61071 \end{bmatrix}, X_2 = \begin{bmatrix} 7.9773 & -2.0683 \\ -2.0683 & 0.60992 \end{bmatrix},$$

defining the designed non-PDC control law:

$$u(t) = \sum_{i=1}^2 h_i(z(t)) F_i \left( \sum_{i=1}^2 h_i(z(t)) X_i \right)^{-1} x(t) \quad (33)$$

which globally stabilizes (30) as illustrated by the phase portrait given in figure 2.

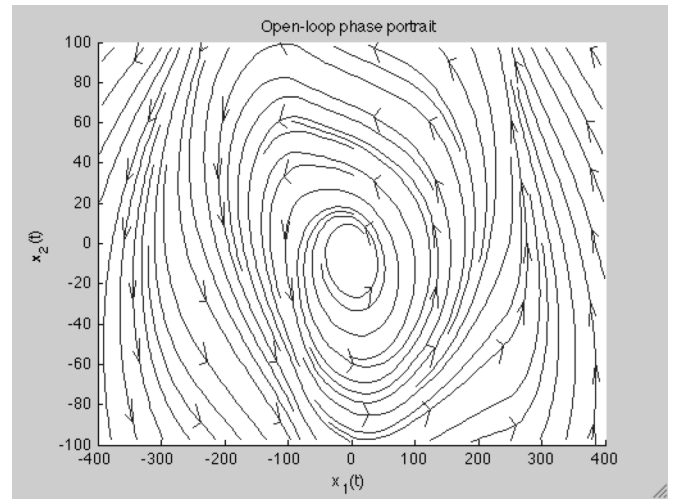


Fig. 1. Open-loop phase portrait.

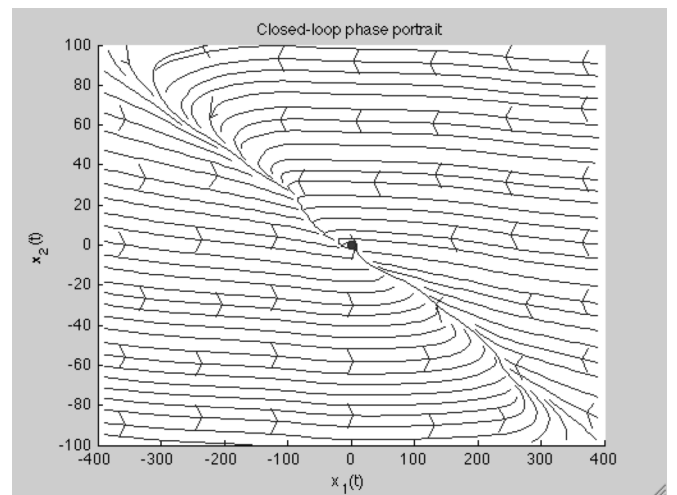


Fig. 2. Closed-loop phase portrait.

## V. CONCLUSION

In this paper, an alternative to LMI approaches for global non-quadratic stabilization of Takagi-Sugeno models has been proposed. Indeed, LMIs have proved their efficiency in fuzzy controller design for Takagi-Sugeno models. However, they suffer from conservatism in the quadratic framework and successive recent advances, such that non-quadratic approaches, lead to complex LMI formulation for a conservatism reduction that may sometimes be questionable. In order to explore what can be done, else than LMIs, SOS based conditions have been proposed for non-quadratic Lyapunov stabilization of TS models. The proposed approach is an attempt as an alternative to LMIs without claiming to make better in terms of conservatism. Having said that, despite most of LMIs approaches, the proposed non-quadratic conditions allow the global stabilization of a TS system on its domain of definition have been obtained. However, the proposed conditions still require some restrictive modeling assumptions and this point, understood as a drawback of SOS formalism, will be the concern of further studies.

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## REFERENCES

- [1] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modelong and control," *IEEE Trans. Syst., Man, Cybern.*, vol. SMC-15, no. 1, pp. 116-132, 1985.
- [2] K. Tanaka and H.O. Wang, "Fuzzy control systems design and analysis: A linear Matrix Inequality Approach," Wiley, New York, USA, 2001.
- [3] K. Tanaka and M. Sugeno, "Stability analysis and design of fuzzy control systems," *Fuzzy Sets and Systems*, vol. 45, no. 2, pp. 135-156, 1992.
- [4] G. Feng, "A survey on analysis and design of model-based fuzzy control systems," *IEEE Trans. Fuzzy Syst.*, vol. 14, no. 5, pp. 676-697, 2006.
- [5] T. Bouarar, K. Guelton, B. Mansouri and N. Manamanni, "LMI stability conditions for Takagi-Sugeno uncertain descriptors," *IEEE International Fuzzy Sytems conference, FUZZ IEEE 2007*, London, UK, July 23-26, 2007.
- [6] Q. Gao, X.J. Zeng, G. Feng, Y. Wang, J. Qiu, "T-S-Fuzzy-Model-Based approximation and controller design for general nonlinear systems," *IEEE Trans. Syst., Man, Cybern.*, vol. 42, no. 4, 1143-1154, 2012.
- [7] A. Sala, "On the conservativeness of fuzzy and fuzzy-polynomial control of nonlinear systems," *Annual Reviews in Control*, vol. 33, no. 1, pp. 48-58, 2009.
- [8] X. Liu, and Q. Zhang, "New approaches to  $H_\infty$  controller design based on fuzzy observers for fuzzy T-S systems via LMI," *Automatica*, vol. 39, no. 9, pp. 1571-1582, 2003.
- [9] H.D. Tuan, P. Apkarian, T. Narikiyo and Y. Yamamoto, "Parameterized linear matrix inequality techniques in fuzzy control system design," *IEEE Trans. Fuzzy Syst.*, vol. 9, no. 2, pp. 324-332, 2001.
- [10] K. Tanaka, T. Hori and H.O. Wang, "A multiple Lyapunov approach to stabilization of fuzzy control systems," *IEEE Trans. Fuzzy Syst.*, vol. 11, no. 4, 582-589, 2003.
- [11] L.A. Mozelli, R.M. Palhares, F.O. Souza and E.M.A.M. Mendes, "Reducing conservativeness in recent stability conditions of TS fuzzy systems," *Automatica*, vol. 46, no. 6, pp. 1580-1583, 2009.
- [12] B.J. Rhee and S. Won, "A new fuzzy Lyapunov function approach for a Takagi-Sugeno fuzzy control system design," *Fuzzy Sets & Systems*, vol. 157, no. 9, pp. 1211-1228, 2006.
- [13] K. Guelton, T.M. Guerra, M. Bernal, T. Bouarar and N. Manamanni, "Comments on fuzzy control systems design via fuzzy Lyapunov functions," *IEEE Trans. Syst., Man, Cybern. B, Cybern.*, vol. 40, no. 3, pp. 970-972, 2010.
- [14] L.A. Mozelli, R.M. Palhares, G.S.C. Avellar, "A systematic approach to improve multiple Lyapunov function stability and stabilization conditions for fuzzy systems," *Information Sciences*, vol. 179, no. 8, pp. 1149-1162, 2009.
- [15] T.M. Guerra, M. Bernal, K. Guelton and S. Labiod, "Non-quadratic local stabilization for continuous-time Takagi-Sugeno models," *Fuzzy Sets and Systems*, vol. 201, pp. 40-54, 2012.
- [16] K. Tanaka, H. Yoshida, H. Othake and H.O. Wang, "A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems," *IEEE Trans. Fuzzy Syst.*, vol. 17, no. 4, pp. 991-922, 2009.
- [17] M. Narimani and H.K. Lam, "SOS-based stability analysis of polynomial fuzzy model-based control systems via polynomial membership functions," *IEEE Trans. Fuzzy Systems*, vol. 18, no. 5, pp. 862-871, 2010.
- [18] K. Guelton, N. Manamanni, D.L. Koumba-Emianiwe and C.D. Chinh, "SOS stability conditions for nonlinear systems based on a polynomial fuzzy Lyapunov function," *18th IFAC World Congress*, vol. 18, no. 1, pp. 12777-12782, Milano, Italy, 2011.
- [19] M. Bernal, A. Sala, A. Jaadari and T.M. Guerra, "Stability analysis of polynomial fuzzy models via polynomial fuzzy Lyapunov functions," *Fuzzy Sets and Systems*, vol. 185, no. 1, pp. 5-14, 2011.
- [20] K. Guelton, N. Manamanni, C.D. Chinh, D.L. Koumba-Emianiwe, "Sum-of-squares stability analysis of Takagi-Sugeno systems based on multiple polynomial Lyapunov functions," *International Journal of Fuzzy Systems*, vol. 15, no. 1, pp. 1-8, 2013.
- [21] T. Bouarar, K. Guelton and N. Manamanni, "Robust fuzzy Lyapunov stabilization for uncertain and disturbed Takagi-Sugeno descriptors," *ISA Transactions*, vol. 49, no. 4, pp. 447-461, 2010.
- [22] S. Prajna, A. Papachristodoulou, P. Seiler and P.A. Parrilo, "SOSTOOLS: Sum of squares optimization toolbox for MATLAB," Version 2.00. California Inst. Technol., Pasadena, 2004.