Geometric-preserving rigid motions of digital objects

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Joint work with

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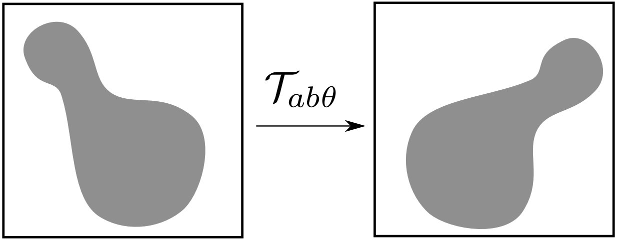
Rigid motion on $\mathbb{R}^2$

A rigid motion is a bijection defined for $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2) \in \mathbb{R}^2$, as

$$\mathcal{T}_{ab\theta}(\mathrm{x}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

with $a, b \in \mathbb{R}$ and $\theta \in [0, 2\pi[$.

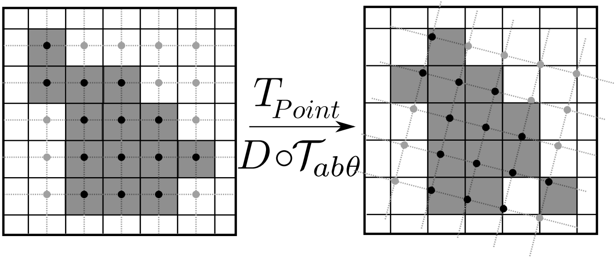


Rigid motion on $\mathbb{Z}^2$

A digital rigid motion on $\mathbb{Z}^2$ is defined for $\mathsf{p} = (\mathsf{p}_1, \mathsf{p}_2) \in \mathbb{Z}^2$ as

$$T_{Point}(\mathsf{p}) = D \circ \mathcal{T}_{ab\theta}(\mathsf{p}) = \begin{pmatrix} [\mathsf{p}_1 \cos\theta - \mathsf{p}_2 \sin\theta + a] \\ [\mathsf{p}_1 \sin\theta + \mathsf{p}_2 \cos\theta + b] \end{pmatrix}$$

where $D : \mathbb{R}^2 \to \mathbb{Z}^2$ is digitization (a rounding function).



Digitized motion and topology preservation

A digital rigid motion on $\mathbb{Z}^2$ is defined for $\mathsf{p} = (\mathsf{p}_1, \mathsf{p}_2) \in \mathbb{Z}^2$ as

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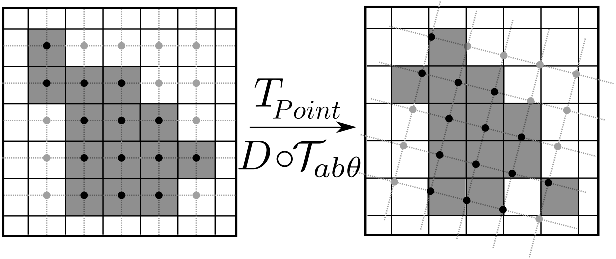
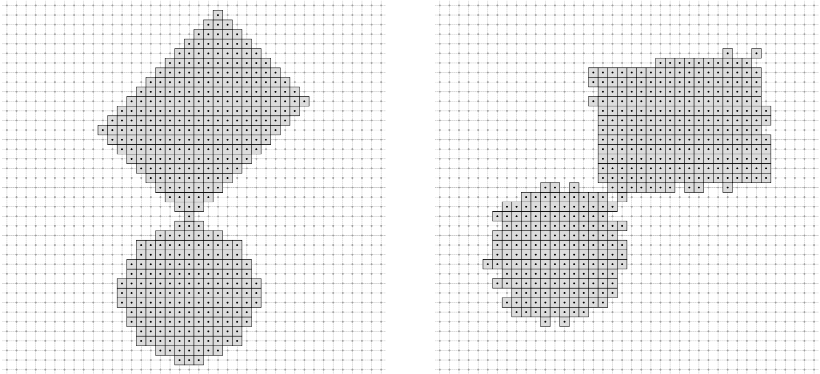
where $D : \mathbb{R}^2 \to \mathbb{Z}^2$ is digitization (a rounding function).

Topology is often altered by digitized rigid motions.



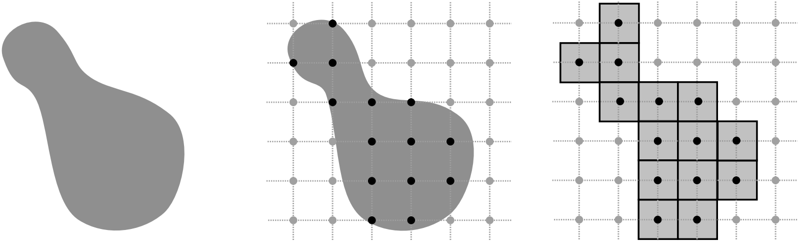
Problem induced by point-wise rigid motion model



Shape and digitization

Given a bounded and connected subset $X \subset \mathbb{R}^2$, its Gauss digitization is defined as:

$$\mathsf{X} = X \cap \mathbb{Z}^2.$$



Digitization and topology preservation

Given a bounded and connected subset $X \subset \mathbb{R}^2$, its Gauss digitization is defined as:

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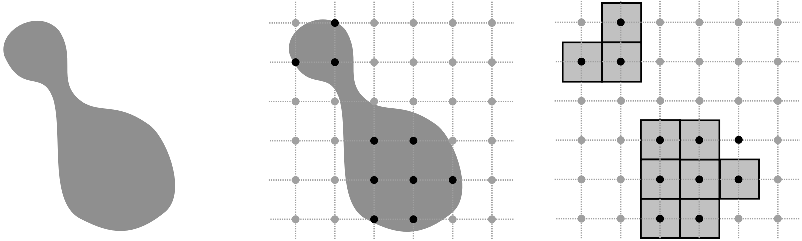
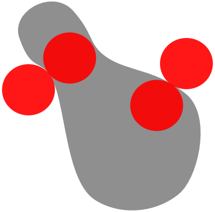
Topology can be altered under the digitization process.

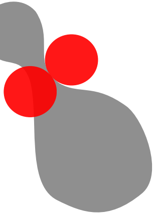


r-regularity for topology preservation

Definition [Pavlidis, 1982]

A bounded and connected subset $X \subset \mathbb{R}^2$ is *r-regular* if for each boundary point of X, there exist two tangent open balls of radius r, lying entirely in X and its complement $\overline{X}$, respectively.





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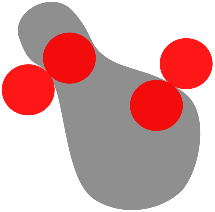
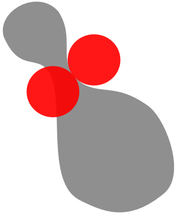


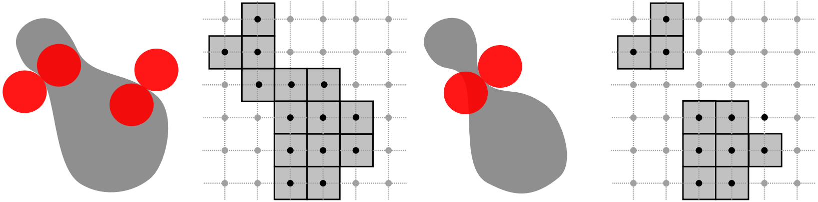


The objects of $\mathbb{R}^2$ with differentiable boundaries.

r-regularity for topology preservation

Proposition [Pavlidis, 1982]

An r-regular set $X \subset \mathbb{R}^2$ has the same topology as its digitized version $\mathsf{X} = X \cap \mathbb{Z}^2$ if $r \geq \frac{\sqrt{2}}{2}$.



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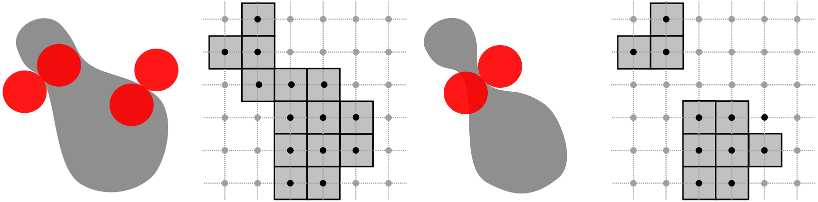
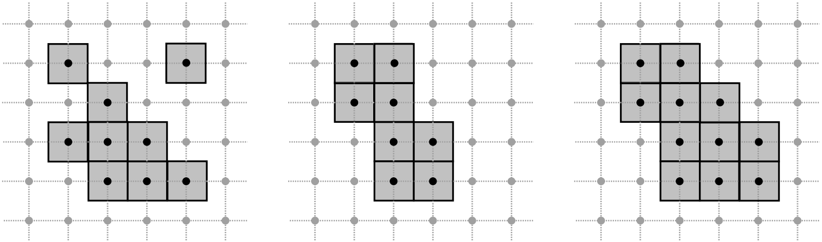


Objects with non-differentiable boundaries (e.g. polygons) ?

Digital regularity for topology preservation

Definition [Ngo et al., 2014]

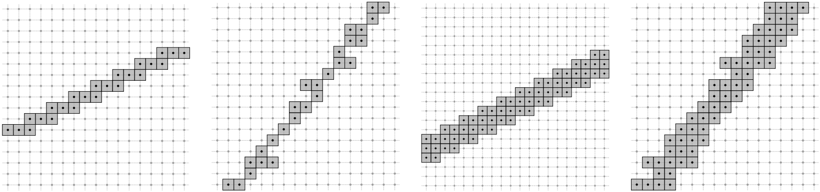
Let $\mathsf{X} \subset \mathbb{Z}^2$ be a well-composed set with no singular point. X is *digitally regular* if for any $\{\mathsf{p}, \mathsf{q}\} \subset \mathsf{X}$ (resp. $\overline{\mathsf{X}}$) of 4-adjacent points, there exists a 2×2 square of points $\{\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{t}\} = \{\mathsf{x}, \mathsf{x} + (0, 1), \mathsf{x} + (1, 0), \mathsf{x} + (1, 1)\}$ s.t $\{\mathsf{p}, \mathsf{q}\} \subset \{\mathsf{x}, \mathsf{y}, \mathsf{z}, \mathsf{t}\} \subseteq \mathsf{X}$ (resp. $\overline{\mathsf{X}}$).



Digital regularity for topology preservation

Proposition [Ngo et al., 2014]

If a well-composed set $X \subset \mathbb{Z}^2$ is digitally regular, then it is topologically invariant under digitized rigid motions.



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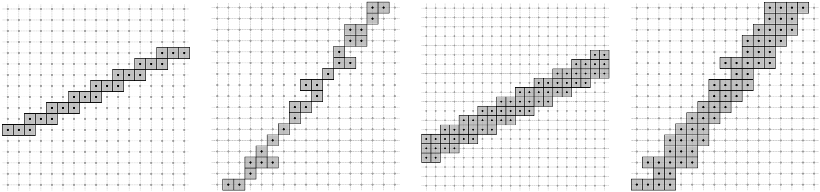


The topology is preserved but not the geometry !

Contributions

- How to preserve the topology of shape whose boundary is non-smooth ?

  $\Rightarrow$ Quasi-r-regularity
- How to perform topological and geometric-preserving rigid motion of digital objets ?

  $\Rightarrow$ Approach via polygonization

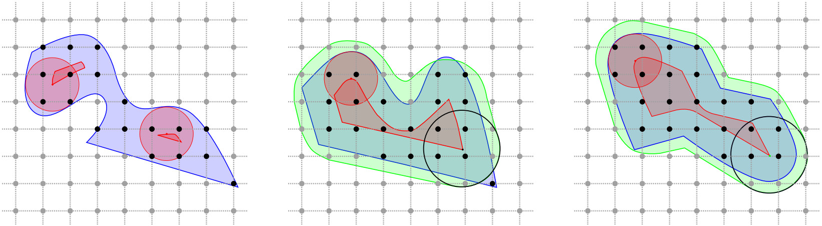
Quasi-r-regularity

Definition [Ngo et al., 2018]

Let $X \subset \mathbb{R}^2$ be a bounded, simply connected set. If

- $X \ominus B_r$ (resp. $\overline{X} \ominus B_r$) is non-empty and connected, and
- $X \subseteq X \ominus B_r \oplus B_{r'}$ (resp. $\overline{X} \subseteq \overline{X} \ominus B_r \oplus B_{r'}$)

for $r' \geq r > 0$, X is *quasi-r-regular* with "margin" $r' - r$.



r-regularity

Definition (in Mathematical Morphology)

Let $X \subset \mathbb{R}^2$ be a finite, simply connected (i.e., connected and wihtout hole) set. If

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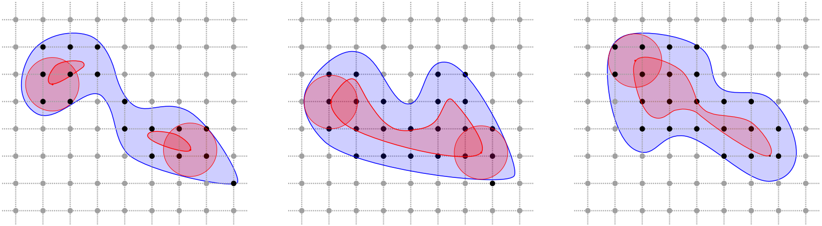
for a given $r > 0$, we say that X is *r-regular*.



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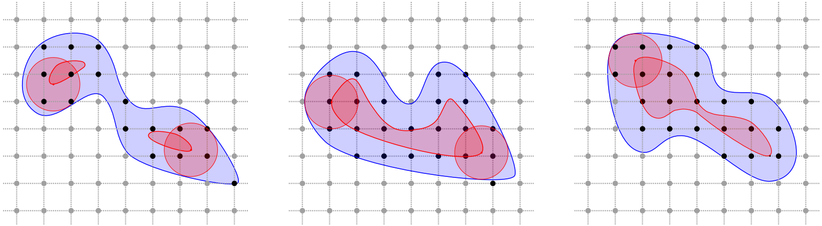
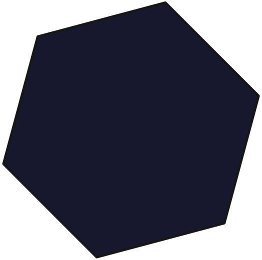


Polygonal objects are not r-regular for any $r > 0$.

Quasi-*r*-regularity for topology preservation

Proposition [Ngo et al., 2018]

If X is quasi-1-regular with margin $\sqrt{2}-1$, then $\mathsf{X} = X \cap \mathbb{Z}^2$ and $\overline{\mathsf{X}} = \overline{X} \cap \mathbb{Z}^2$ are both 4-connected. In particular, X is then well-composed.



Quasi-r-regularity for topology preservation

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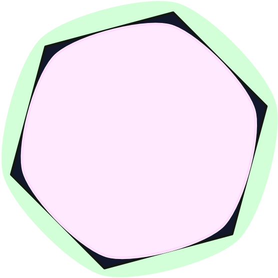

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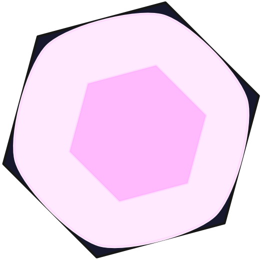
Idea of proof:

- $X \circ B_1 = X \ominus B_1 \oplus B_1$ is 1-regular, then $(X \circ B_1) \cap \mathbb{Z}^2$ is 4-connected.



Quasi-r-regularity for topology preservation

Proposition [Ngo et al., 2018]

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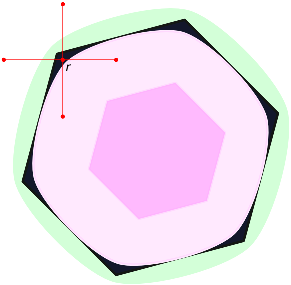
Idea of proof:

- $X \circ B_1 = X \ominus B_1 \oplus B_1$ is 1-regular, then $(X \circ B_1) \cap \mathbb{Z}^2$ is 4-connected.
- With any position of $\mathbb{Z}^2$, if there exists $\mathsf{r} \in \mathbb{Z}^2$ in $X \setminus (X \circ B_1)$, then r is 4-adjacent to a point of $(X \circ B_1) \cap \mathbb{Z}^2$.



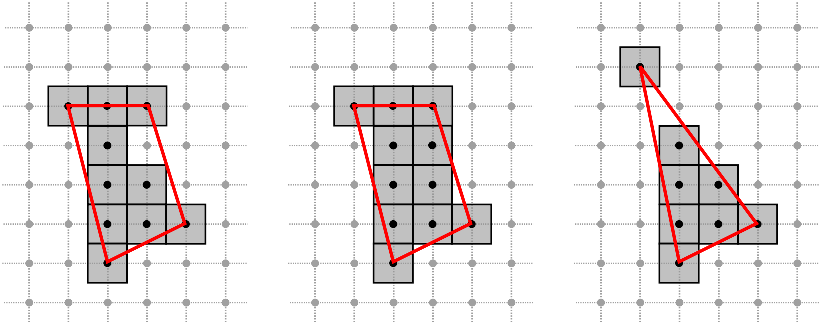
H-convexity

Definition [Kim, 1981]

A digital object $\mathsf{X} \subset \mathbb{Z}^2$ is H-convex if

$$\mathsf{X} = \mathcal{C}onv(\mathsf{X}) \cap \mathbb{Z}^2$$

where $\mathcal{C}onv(\mathsf{X})$ is the convex hull of X.

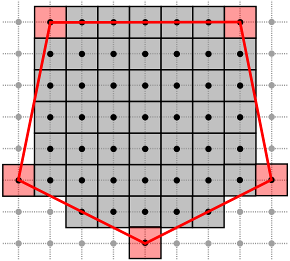


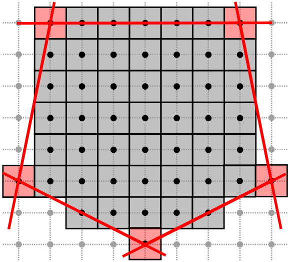
Half-plane representation of H-convex object

Let X be a H-convex object containing at least three non-colinear points, $\mathcal{C}onv(\mathsf{X})$ be the convex hull of X. Then,

$$\mathsf{X} = \mathcal{C}onv(\mathsf{X}) \cap \mathbb{Z}^2 = \Big(\bigcap_{\mathrm{H} \in \mathcal{R}(\mathsf{X})} \mathrm{H} \Big) \cap \mathbb{Z}^2 = \bigcap_{\mathrm{H} \in \mathcal{R}(\mathsf{X})} \left(\mathrm{H} \cap \mathbb{Z}^2 \right)$$

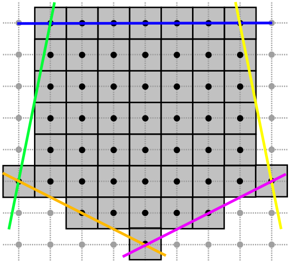
where $\mathcal{R}(\mathsf{X})$ is the minimal set of closed half-planes that include X. Each closed half-plane H has coefficients defined by two consecutive vertices of $\mathcal{C}onv(\mathsf{X})$.

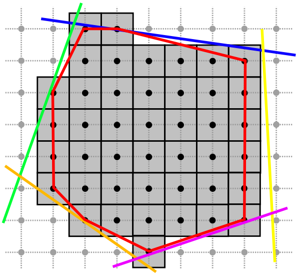




Rigid motion of H-convex object via convex hull

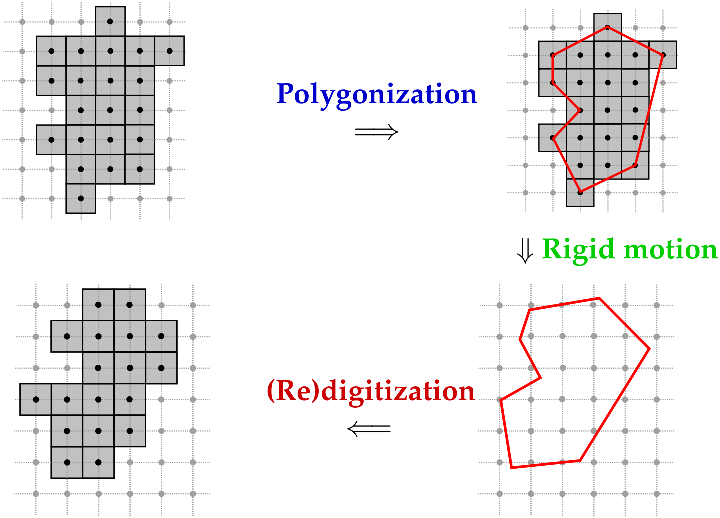
$$T_{\mathcal{C}onv}(\mathsf{X}) = \mathcal{T}(\mathcal{C}onv(\mathsf{X})) \cap \mathbb{Z}^2 = \mathcal{T}\left(\bigcap_{\mathrm{H}\in\mathcal{R}(\mathsf{X})} \mathrm{H}\right) \cap \mathbb{Z}^2$$





$$\mathcal{C}onv(T_{\mathcal{C}onv}(\mathsf{X})) \subseteq \mathcal{T}(\mathcal{C}onv(\mathsf{X}))$$

Rigid motion non-convex object via polygonization



$$T_{\mathcal{P}oly}(\mathsf{X}) = \mathcal{T}(\mathcal{P}oly(\mathsf{X})) \cap \mathbb{Z}^2$$

Topological and geometric-preserving rigid motions

Proposition

Let X be a digital object and $T_{\mathcal{C}onv}$ be the rigid motion induced by a rigid motion $\mathcal{T}$. If X is H-convex, then $T_{\mathcal{C}onv}(\mathsf{X})$ is H-convex.

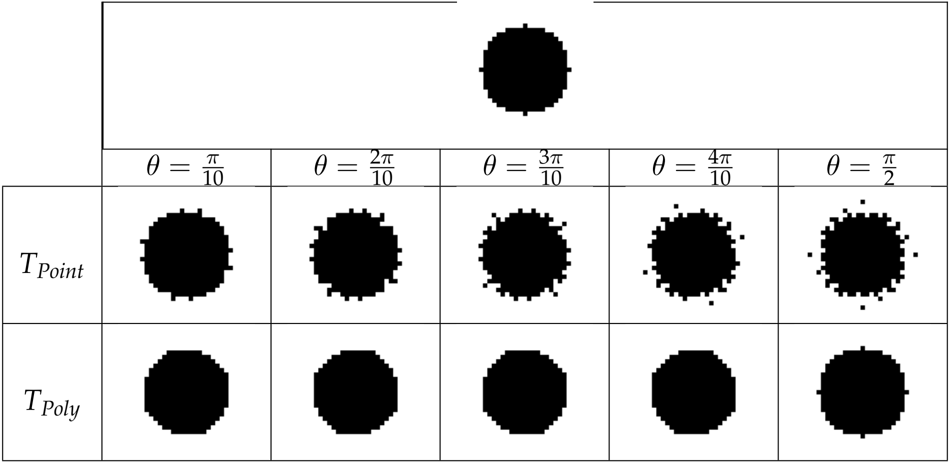
Proposition

Let X be an H-covnex digital object. If $\mathcal{C}onv(\mathsf{X})$ is quasi-1-regular with margin $\sqrt{2}-1$, then $T_{\mathcal{C}onv}(\mathsf{X})$ is well-composed.

Proposition

Let $\mathsf{X} \subset \mathbb{Z}^2$ be a digital object. Let $P(\mathsf{X}) \subset \mathbb{R}^2$ be a polygon such that $P(\mathsf{X}) \cap \mathbb{Z}^2 = \mathsf{X}$. If $P(\mathsf{X})$ is quasi-1-regular with margin $\sqrt{2}-1$, then $T_{\mathcal{P}oly}(\mathsf{X})$ is well-composed.

Experimental results



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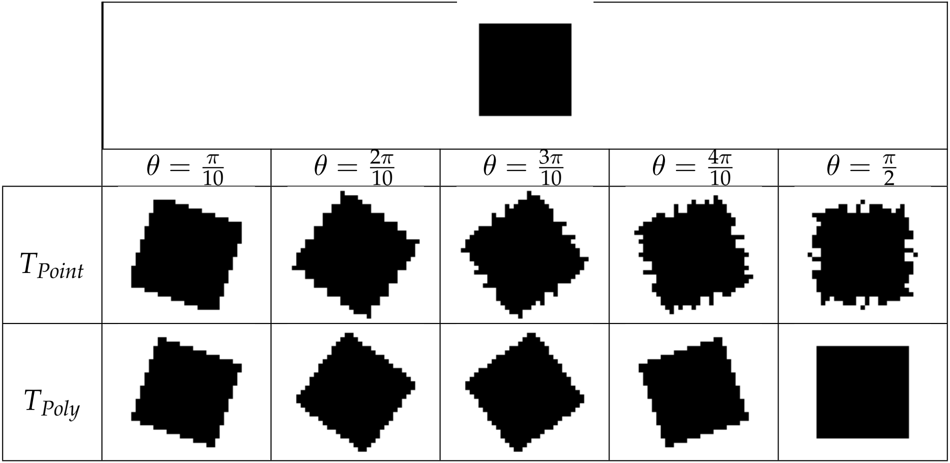

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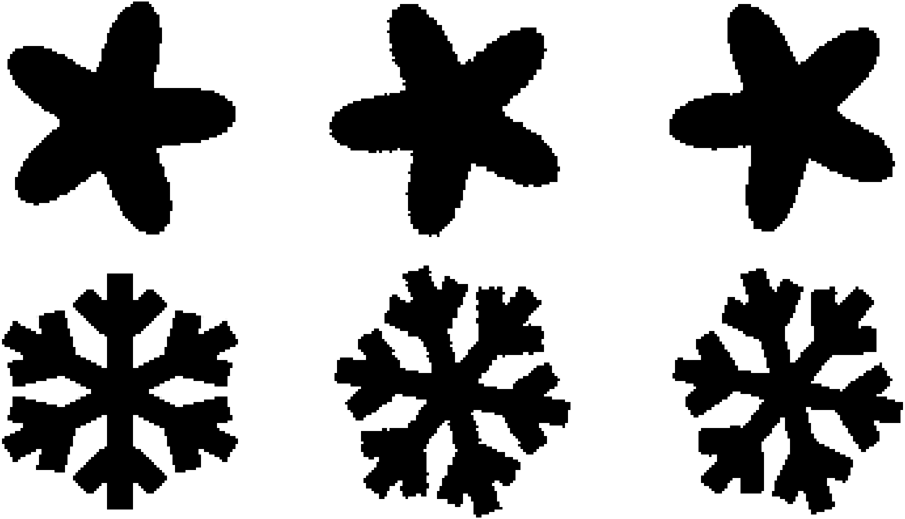


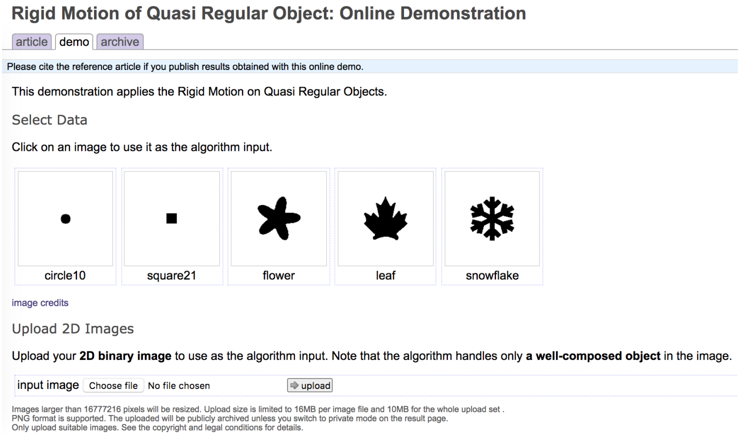
X $T_{Point}(X)$ $T_{\mathcal{P}oly}(X)$

Experimental results

X $T_{Point}(X)$ $T_{\mathcal{P}oly}(X)$

Online Demonstration

An online demonstration based on the DGtal library, is available at the following website:
`http://ipol-geometry.loria.fr/~phuc/ipol_demo/RigidMotion2D`



Conclusion

Contributions:

- A sufficient condition, namely *quasi-regularity*, for preserving the topology and certain geometric properties during the Gaussian digitization.
- A rigid motion scheme based on polygonal representation that preserves geometry and topology properties of the transformed digital object.

Perspectives:

- Geometric characterization of quasi-regularity.
- A polygonalization method providing quasi-regular polygons of digital objects.
- Regularization method for non quasi-regular polygons.

Thank you for your attention!

References I

Kim, C. E. (1981).
On the cellular convexity of complexes.
IEEE Transactions on Pattern Analysis and Machine Intelligence, PAMI-3(6):617–625.

Lien, J.-M. and Amato, N. M. (2006).
Approximate convex decomposition of polygons.
Comput. Geom. Theory Appl., 35(1-2):100–123.

Ngo, P., Passat, N., Kenmochi, Y., and Debled-Rennesson, I. (2018).

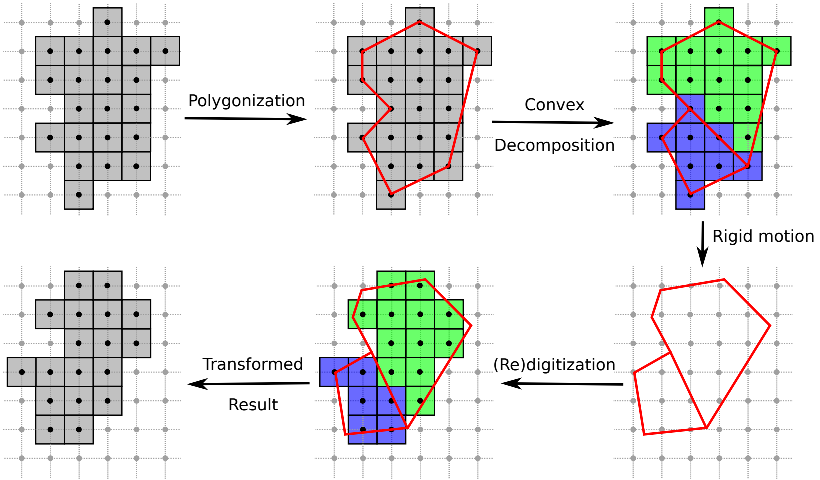
Geometric preservation of 2d digital objects under rigid motions.
Submitted to Journal of Mathematical Imaging and Vision.

References II

Ngo, P., Passat, N., Kenmochi, Y., and Talbot, H. (2014).
Topology-preserving rigid transformation of 2D digital images.
IEEE Transactions on Image Processing, 23(2):885–897.

Pavlidis, T. (1982).
Algorithms for graphics and image processing.
Berlin: Springer, and Rockville: Computer Science Press.

Rigid motion non-convex object via polygonization

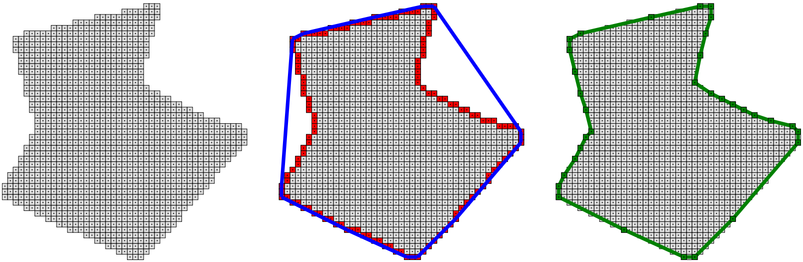


Polygonization of digital objects

The method is based contour points and the convex hull

1. Extract 8-connected contour points of X
2. Compute convex hull of X
3. Determine the segments that best fit the concave parts of X

$$X = P(X) \cap \mathbb{Z}^2$$

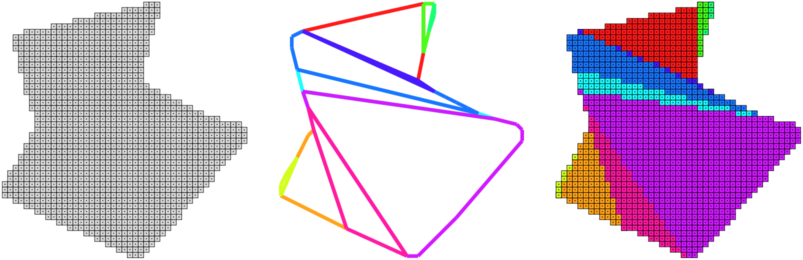


Convex decomposition of polygons

The method [Lien and Amato, 2006] decomposes a simple polygon into convex pieces by iteratively removing the most significant non-convex features.

$$P = \bigcup P_i$$

$$\mathsf{X} = P(\mathsf{X}) \cap \mathbb{Z}^2 = \bigcup \left(P_i \cap \mathbb{Z}^2\right).$$



Extension to 3D

Definition

Let $X \subset \mathbb{R}^3$ be a bounded, simply connected set. If

- $X \ominus B_r$ (resp. $\overline{X} \ominus B_r$) is non-empty and connected, and
- $X \subseteq X \ominus B_r \oplus B_{r'}$ (resp. $\overline{X} \subseteq \overline{X} \ominus B_r \oplus B_{r'}$)

for $r' \geq r > 0$, X is *quasi-r-regular* with "margin" $r' - r$.

Proposition

Let $\mathsf{X} \subset \mathbb{Z}^3$ be a digital object. If X is quasi-1-regular with margin $\frac{2}{\sqrt{3}} - 1$, then $\mathsf{X} = X \cap \mathbb{Z}^3$ and $\overline{\mathsf{X}} = \overline{X} \cap \mathbb{Z}^3$ are both 6-connected.

Experimental results