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Issam Chekakta * **Kevin Guelton** ** **Dalel Jabri** *
Djamel E.C. Belkhiat * **Koffi M.D. Motchon** **

* *DAC-HR Laboratory, Ferhat Abbas University Setif-1, Setif, Algeria (e-mail: {issam.chekakta,dalel.jabri,djamel.belkhiat}@univ-setif.dz).*
** *CReSTIC EA3804, Université de Reims Champagne-Ardenne, Campus Moulin de la Housse BP1039, 51687 Reims Cedex 2, France (e-mail: {kevin.guelton,koffi.motchon}@univ-reims.fr).*

Abstract: This paper presents a model reference attitude tracking controller design for a quadrotor unmanned aerial vehicle subject to saturated actuators and external disturbances. The dynamical nonlinear model of the quadrotor's attitude is represented by an uncertain-like Takagi-Sugeno model, with exact matching on a compact subset of the state space. The generalized sector condition is employed to deal with the saturated inputs of the quadrotor. LMI conditions are derived for the design of the proposed tracking controller from a quadratic Lyapunov function, together with a performance index used to minimize the L_2-norm transfer between the disturbances and the state tracking errors. Because of the input saturation and the validity domain of the Takagi-Sugeno model, the obtained results only hold regionally and an optimization procedure to estimate the closed-loop tracking domain of attraction is proposed. Simulation results are provided to illustrate the effectiveness of the proposed design methodology.

Keywords: Quadrotor UAVs, Regional attitude tracking control, Takagi-Sugeno models, Input saturation, LMIs.

1. INTRODUCTION

Unmanned Aerial Vehicles (UAVs) are now often used in the military and civic sectors, as well as in transportation, agriculture, and in a variety of other fields. Quadrotors are highly popular in these situations because of their lightweight, simple structure, and maneuverability. Therefore, researchers have focused on developing mathematical tools enabling these nonlinear systems to fly autonomously, see e.g (Bouabdallah, 2007; Maya-Gress et al., 2021). Because of the large number of tools available in linear control, linear controllers were the most commonly used controllers for such systems. For instance, a Robust PID controller was proposed by Miranda-Colorado and Aguilar (2020) to stabilize and track the reference trajectory by optimizing the controller gains through a cuckoo search technique; the LQR/LQG theory was used to construct a MIMO PID controller for quadrotor trajectory tracking in Guardeño et al. (2019). However, it is worth noticing that such linear control strategies only guarantee the stability of a quadrotor when its attitude stay close to the hover position. Therefore, when aggressive maneuver are required, nonlinear control should be considered for the attitude control loop.

Among nonlinear control approaches, Takagi-Sugeno (T-S) fuzzy models (Takagi and Sugeno, 1985) have attracted the attention of the control community due to their ability to match nonlinear systems precisely by expanding some control principles that were initially dedicated to linear systems. As a result, various theoretical articles have been published, using T-S models for controllers and observers design to several classes of control problems; e.g. quadratic design (Tanaka and Wang, 2001), switched T-S systems (Belkhiat et al., 2019; Chekakta et al., 2021), output feedback control (Zerar et al., 2008; Jabri et al., 2020), T-S descriptor control (Taniguchi et al., 2000; Schulte and Guelton, 2009), tracking control with external disturbances attenuation from an H_∞ performance index (Mansouri et al., 2009; Seddiki et al., 2010), and so on. Aside these works, many authors have used T-S modeling to represent nonlinear quadrotor dynamics for control purpose. In this context, a fixed quadrotor controller and observer were constructed, with experimental results in Lendek et al. (2013). In addition, Yacef (2012) built a nominal three-rules T-S model for the quadcopter with Taylor series expansion. Moreover, in Cherifi et al. (2018), a D-stability robust controller was synthesized to stabilize the attitude of a quadrotor. In a similar context, in Torres et al. (2016) and in Sheikhpour and Shouraki (2013), a fuzzy state feedback controller is proposed based on experimental and simulation results. A state feedback controller with all six degrees of freedom (position and attitude) was developed in Pedro and Kala (2015). Relying on the work of Cherifi et al. (2018), a T-S model of the quadrotor is used where nonlinear consequent part are taking into account as structured uncertainties.

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Model reference tracking control is an important feature for many practical applications, especially for the attitude control loop when quadrotors' aggressive maneuver are involved. A reference model is introduced in the closed-loop dynamics to improve tracking performances, despite the complexity of the trajectory, impacting the system's dynamics to improve accuracy and robustness according to the desired dynamics (Mansouri et al., 2009; Seddiki et al., 2010). Moreover, in many practical application, input saturation occurs because actuators are usually physically limited. To cope with such input constraints, many works have been done in the linear framework (Tarbouriech et al., 2011), or in the T-S framework, see e.g. (Nguyen et al., 2016; Lopes et al., 2020, 2021). However, because of the domain of validity of the T-S model and the limited input signals, such designed tracking controllers would be only suitable regionally.

Based on the above considerations, it is worth noticing that, to the best of the authors knowledge, there is no previous studies from the literature, which explore the regional T-S model-based attitude tracking control of quadrotors subject to actuators' saturation and external disturbances, with the analysis of the closed-loop domain of attraction. This paper aims at dealing with this issue, i.e. providing Linear Matrix Inequality (LMI) conditions as a design methodology of T-S model-based attitude tracking controllers for disturbed quadrotors involving input constraints.

The remainder of this paper is organized as follows. Section 2 presents the mathematical model of the quadrotor with external disturbances and actuators' saturation, as well as the problem statement. In section 3, LMI-based conditions for the design of the proposed model reference-based attitude tracking control scheme is presented with an optimization procedure to estimate the closed-loop domain of attraction with regards to the validity domain of the T-S model and the input constraints. Finally, simulation results of the designed closed-loop tracking quadrotor attitude dynamics are provided to illustrate the effectiveness of the proposed control approach.

Notations. The notations used in this paper are standard, a star $(*)$ in a matrix denotes a transpose quantity; we denote the finite set of integers $\mathcal{I}_r = \{1, ..., r\}$; $\mathcal{H}(M)$ is a shorthand of $M + M^T$, $M_{(l)}$ denotes the l^{th} line of M; moreover, convex combinations of matrices M_i ($i \in \mathcal{I}_r$) with appropriate dimensions are denoted as $M_h = \sum_{i=1}^{r} h_i M_i$.

2. PRELIMINARIES AND PROBLEM STATEMENT

By extending quadrotor's modelling proposed in (Bouabdallah, 2007), let us consider the attitude dynamics of a quadrotor, affected by actuators saturation and external disturbances, described by:

$$\begin{cases} \ddot{\phi}(t) = \frac{J_r\dot{\theta}(t)}{I_{xx}}u_g(t) + I_{yzx}\dot{\theta}(t)\dot{\psi}(t) + \frac{\text{sat}(u_1(t))}{I_{xx}} + \varphi_1(t) \\ \ddot{\theta}(t) = \frac{-J_r\dot{\phi}(t)}{I_{yy}}u_g(t) + I_{zxy}\dot{\phi}(t)\dot{\psi}(t) + \frac{\text{sat}(u_2(t))}{I_{yy}} + \varphi_2(t) \\ \ddot{\psi}(t) = I_{xyz}\dot{\phi}(t)\dot{\theta}(t) + \frac{\text{sat}(u_3(t))}{I_{zz}} + \varphi_3(t) \end{cases} \quad (1)$$

where $\phi(t)$, $\theta(t)$ and $\psi(t)$ denote respectively the roll, pitch and yaw angles, $\dot{\phi}(t)$, $\dot{\theta}(t)$ and $\dot{\psi}(t)$ represent their angular velocities, for $i \in \mathcal{I}_3$, $u_i(t)$, are the attitude torque control inputs, $u_g(t) = -w_1(t) + w_2(t) - w_3(t) + w_4(t)$ is the gyroscopic effect depending on the rotors' angular velocities w_j ($j \in \mathcal{I}_4$), $\varphi_i(t)$ are the external disturbances, assumed to belong to $L_2[0, +\infty)$. Moreover, we denotes $I_{yzx} = \frac{I_{yy}-I_{zz}}{I_{xx}}$, $I_{xyz} = \frac{I_{xx}-I_{yy}}{I_{zz}}$ and $I_{zxy} = \frac{I_{zz}-I_{xx}}{I_{yy}}$, with the inertial parameters given in Table 2.

Table 1. Quadrotor parameters (Jeurgens, 2017)

Symbols	Value	Unit	Description
l	0.178	m	Distance from the center of gravity to the rotors
I_x	$2.23 \cdot 10^{-3}$	$kg \cdot m^2$	Moment of inertia for x-axes
I_y	$2.98 \cdot 10^{-3}$	$kg \cdot m^2$	Moment of inertia for y-axes
I_z	$4.80 \cdot 10^{-3}$	$kg \cdot m^2$	Moment of inertia for z-axes
I_r	$2.029 \cdot 10^{-5}$	$kg \cdot m^2$	Moment of inertia of each rotor
b	$1.7231 \cdot 10^{-6}$	N	Thrust Coefficient
d	$2.2169 \cdot 10^{-7}$	$N \cdot m$	Drag Coefficient

In this paper, it is assumed that input saturation occurs, that is to say, $\forall i = \mathcal{I}_3$, $|u_i(t)| \leq \bar{u}_i$, and:

$$\text{sat}(u_i(t)) = \text{sign}(u_i(t)) \min\{|u_i(t)|, \bar{u}_i\} \quad (2)$$

Let us recall that the input torques $u_i(t)$ result from the combination of 4 rotor forces $F_j = b\varpi_j^2$ (for $j \in \mathcal{I}_4$), assuming the X quadrotor configuration, where b is a thrust constant. In this context, the following application define the relation between the input signals $u_i(t)$ ($i \in \mathcal{I}_3$) and the *pwm* commands ϖ_j ($j \in \mathcal{I}_4$) (Jeurgens, 2017):

$$\begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{bmatrix} = \begin{bmatrix} \frac{bl}{\sqrt{2}} & \frac{-bl}{\sqrt{2}} & \frac{-bl}{\sqrt{2}} & \frac{bl}{\sqrt{2}} \\ \frac{-bl}{\sqrt{2}} & \frac{-bl}{\sqrt{2}} & \frac{bl}{\sqrt{2}} & \frac{bl}{\sqrt{2}} \\ \frac{-d}{b} & \frac{d}{b} & \frac{-d}{b} & \frac{d}{b} \end{bmatrix} \begin{bmatrix} w_1^2(t) \\ w_2^2(t) \\ w_3^2(t) \\ w_4^2(t) \end{bmatrix}, \varpi_j(t) = \frac{w_j(t) - \kappa_b}{\kappa_a}, \quad (3)$$

where $\kappa_a = 3.71$ and $\kappa_b = 138.8$ $rad.s^{-1}$ are parameters of the linear relation between each rotor angular velocity $w_j(t)$ and the *pwm* (pulse width modulation) motor commands $\varpi_j(t) \in [0, 100]$ (%), l is the length from the center of the UAV to the rotors and d is the drag coefficient.

Let $x(t) = \left[\phi(t), \theta(t), \psi(t), \dot{\phi}(t), \dot{\theta}(t), \dot{\psi}(t)\right]^T$ and $u(t) = [u_1(t), u_2(t), u_3(t)]^T$ be respectively the state and input vectors. The nonlinear dynamics (1) can be written as the following affine-in-control nonlinear state space model:

$$\dot{x}(t) = A(x(t), u_g(t))x(t) + B\text{sat}(u(t)) + W\varphi(t) \quad (4)$$

with:

$$A(x(t), u_g(t)) = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{J_r u_g(t)}{I_{xx}} & I_{yzx}x_5(t) \\ 0 & 0 & 0 & -\frac{J_r u_g(t)}{I_{yy}} & 0 & I_{zxy}x_4(t) \\ 0 & 0 & 0 & I_{xyz}x_5(t) & 0 & 0 \end{bmatrix},$$

$$B^T = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{I_{xx}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{I_{yy}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{I_{zz}} \end{bmatrix}, W^T = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Thanks to the sector nonlinearity approach (Tanaka and Wang, 2001), a T-S model matching exactly (4) can be obtained. However, note that (4) contains a nonlinear entry $u_g(t)$, which depends on the input variables. Therefore, for control purpose and to avoid algebraic loop for practical implementation, such nonlinear term should be removed from the fuzzy membership functions. To cope with that issue, since we have $|u_g(t)| \leq \bar{u}_g$, Cherifi et al.

(2018) propose a T-S modeling approach with nonlinear consequent part, where the terms depending on $u_g(t)$ are treated like structural uncertainties. To do so, let us rewrite $A(x(t), u_g(t))$ as:

$$A(x(t), u_g(t)) = \mathcal{A}(x(t)) + \Delta\mathcal{A}(u_g(t)) \tag{5}$$

where $\Delta\mathcal{A}(u_g(t)) = \begin{bmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&\frac{J_r u_g(t)}{I_{xx}}&0\\ 0&0&0&-\frac{J_r u_g(t)}{I_{yy}}&0&0\\ 0&0&0&0&0&0 \end{bmatrix} = H\delta(t)E_a$

with $\delta(t) = \frac{u_g(t)}{\max(u_g(t))}$ satisfying $\delta^2(t) \leq 1$, $H = \begin{bmatrix} 0&0&0&0&1&0\\ 0&0&0&1&0&0 \end{bmatrix}^T$ and $E_a = \begin{bmatrix} 0&0&0&\frac{-\bar{u}_g J_r}{I_{yy}}&0&0\\ 0&0&0&0&\frac{\bar{u}_g J_r}{I_{xx}}&0 \end{bmatrix}$.

Now, assuming $|x_4(t)| \leq \bar{x}_4$ and $|x_5(t)| \leq \bar{x}_5$ (defining a compact set of the state space $\mathcal{D}_x \subset \mathbb{R}^6$), then applying the sector nonlinearity approach (see Tanaka and Wang (2001)) on $\mathcal{A}(x(t)) = \mathcal{A}(x_4(t), x_5(t)) = A(x(t), u_g(t)) - \Delta\mathcal{A}(u_g(t))$ with the vector of premises $z(t) = [x_4(t)\ x_5(t)]^T$, we obtain the following uncertain-like T-S model with r=4 vertices, which exactly match (4) on $\mathcal{D}_x$.

$$\dot{x}(t) = \sum_{i=1}^{4} h_i(z(t))(\mathcal{A}_i + \Delta\mathcal{A}(u_g(t)))x(t) + B\text{sat}(u(t)) + W\varphi(t) \tag{6}$$

with the matrices $\mathcal{A}_1 = \mathcal{A}(-\bar{x}_4, -\bar{x}_5)$, $\mathcal{A}_2 = \mathcal{A}(-\bar{x}_4, \bar{x}_5)$, $\mathcal{A}_3 = \mathcal{A}(\bar{x}_4, -\bar{x}_5)$, $\mathcal{A}_4 = \mathcal{A}(\bar{x}_4, \bar{x}_5)$ and the positive membership functions $h_1(z(t)) = (\bar{x}_4 - x_4(t))(\bar{x}_5 - x_5(t))/4\bar{x}_4\bar{x}_5$, $h_2(z(t)) = (\bar{x}_4 - x_4(t))(x_5(t) + \bar{x}_5)/4\bar{x}_4\bar{x}_5$, $h_3(z(t)) = (x_4(t) + \bar{x}_4)(\bar{x}_5 - x_5(t))/4\bar{x}_4\bar{x}_5$, $h_4(z(t)) = (x_4(t) + \bar{x}_4)(x_5(t) + \bar{x}_5)/4\bar{x}_4\bar{x}_5$, which satisfy the convex sum properties $\sum_{i=1}^{4} h_i(z(t)) = 1$.

Let us now consider the tracking control scheme depicted in Fig. 1 with a linear reference model given by:

$$\dot{x}_r(t) = A_r x_r(t) + B_r r(t) \tag{7}$$

with $A_r \in \mathbb{R}^{6\times 6}$ a Hurwitz matrix, $x_r(t) \in \mathbb{R}^6$ the reference state vector and $r(t) \in \mathbb{R}^m$ $(m \leq 6)$ the desired trajectory to be tracked by the reference model.

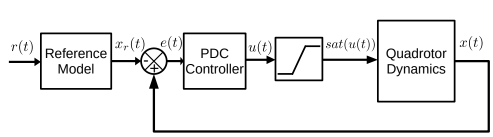

Fig. 1. Quadrotor Attitude Control Strategy

We consider a tracking PDC control law given by:

$$u(t) = \sum_{i=1}^{4} h_i(z(t))K_i\big(x(t) - x_r(t)\big) \tag{8}$$

Let us define the function $\Psi(u(t)) = \text{sat}(u(t)) - u(t)$, the extended state vector $\tilde{x}(t) = \left[x^T(t) - x_r^T(t)\ x_r^T(t)\right]^T$ and extended external disturbance vector $\omega(t) = \left[\varphi^T(t)\ r^T(t)\right]^T$, the closed loop dynamics can be written as:

$$\dot{\tilde{x}}(t) = (\tilde{A}_h + \tilde{B}\tilde{K}_h)\tilde{x}(t) + \tilde{B}\Psi(u(t)) + \tilde{F}\omega(t) \tag{9}$$

with $\tilde{A}_h = \begin{bmatrix} \mathcal{A}_h + \Delta\mathcal{A} & \mathcal{A}_h - A_r + \Delta\mathcal{A}\\ 0 & A_r \end{bmatrix}$, $\tilde{K}_h^T = \begin{bmatrix} K_h\\ 0 \end{bmatrix}^T$, $\tilde{B} = \begin{bmatrix} B\\ 0 \end{bmatrix}$ and $\tilde{F} = \begin{bmatrix} W & -B_r\\ 0 & B_r \end{bmatrix}$.

It is worth noticing that, with the considered extended state vector, the domain of validity of the considered closed-loop T-S model can be rewritten as:

$$\mathcal{D}_{\tilde{x}} = \{\tilde{x} \in \mathbb{R}^{12} : |\mathfrak{L}\tilde{x}(t)| \leq \mathcal{Q}\} \tag{10}$$

with $\mathfrak{L} = \begin{bmatrix} 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&-1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&-1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&-1&0 \end{bmatrix}$ and $\mathcal{Q} = \begin{bmatrix} 2\bar{x}_4\\ 2\bar{x}_4\\ 2\bar{x}_5\\ 2\bar{x}_5\\ \bar{x}_{r4}\\ \bar{x}_{r4}\\ \bar{x}_{r5}\\ \bar{x}_{r5} \end{bmatrix}$.

It should be highlighted that the T-S model (6) is valid and guaranteed to be an exact polytopic convex representation of the nonlinear model (1) inside the domain of validity $\mathcal{D}_x$, according to the premise variables defined above (6) and their bounds $|x_4(t)| \leq \bar{x}_4$ and $|x_5(t)| \leq \bar{x}_5$. Therefore, integrating a reference model in the design leads to the construction of the domain of validity for the reference model $\mathcal{D}_{x_r}$ with the same bounds $|x_{r,4}(t)| \leq \bar{x}_4$ and $|x_{r,5}(t)| \leq \bar{x}_5$. Furthermore, from $\mathcal{D}_x$ and $\mathcal{D}_{x_r}$ we can get the domain of validity of the tracking error $e(t) = x(t) - x_r(t)$ denoted as $\mathcal{D}_e$ with $|x_4(t) - x_{r,4}(t)| \leq 2\bar{x}_4$ and $|x_5(t) - x_{r,5}(t)| \leq 2\bar{x}_5$. Hence, the closed-loop dynamics (9) belong to the domain of validity $\mathcal{D}_{\tilde{x}} = \mathcal{D}_e \times \mathcal{D}_{x_r}$. This explains the size and entries of $\mathfrak{L}$ and $\mathcal{Q}$ given in (10).

Moreover, because of the occurrence of input saturation, the following lemma will be used to characterize the input operation domain $\mathcal{D}_u$.

Lemma 1. (Tarbouriech et al., 2011) Given two matrices $K_i \in \mathbb{R}^{m\times n}$ and $\mathcal{T}_i \in \mathbb{R}^{m\times n}$, let $\mathcal{D}_u$ be the polyhedral set related with these matrices and defined by:

$$\mathcal{D}_u = \left\{\tilde{x} \in \mathbb{R}^{2\text{n}} : \left|\left[(K_{i(s)} - \mathcal{T}_{i(s)})\ 0_{1\times n}\right]\tilde{x}\right| \leq \bar{u}_{(s)}\right\}, s \in \mathcal{I}_m \tag{11}$$

Then, the nonlinearity $\Psi(u)$ satisfies the inequality:

$$\Psi(u)^T \sum_{i=1}^{r} h_i \Upsilon_i \big(\Psi(u) - \sum_{i=1}^{r} h_i \mathcal{T}_i(x - x_r)\big) \leq 0 \tag{12}$$

for any $m \times m$ positive diagonal matrix Υ_i.

Problem Statement. Synthesize the gain matrices of the tracking PDC controller (8) such that:

i. The closed loop tracking dynamics is regionally asymptotically stable when $\omega(t) = 0$ (i.e. $\phi(t) = 0$ and $r(t) = 0$), with an estimate of the closed-loop domain of attraction $\mathcal{D}_a \subseteq \mathcal{D}_{\tilde{x}} \cap \mathcal{D}_u$.

ii. For all non zero $\omega(t) \in L_2[0, \infty)$, the L_2-norm transfer $\gamma > 0$ between $\omega(t)$ and $x(t) - x_r(t)$ is minimized, i.e.:

$$\min \gamma, \text{ s.t. } \frac{||x(t) - x_r(t)||_2}{||\omega((t)||_2} < \gamma \tag{13}$$

or equivalently, with $\tilde{Q} = \text{diag}(I, 0)$:

$$\min \gamma, \text{ s.t. } \int_0^\infty \tilde{x}^T(t)\tilde{Q}\tilde{x}(t)dt < \gamma^2 \int_0^\infty \omega^T(t)\omega(t)dt \tag{14}$$

To achieve these goals, the following lemma will be employed to obtain the main result in the next section.

Lemma 2. (Mansouri et al., 2009) For real matrices A, B, W, Y, Z and a regular matrix M with appropriate dimensions one has:

$$\begin{bmatrix} Y + B^{\text{T}}M^{-1}B & (*)\\ W & Z + AMA^{\text{T}} \end{bmatrix} < 0 \Rightarrow \begin{bmatrix} Y & (*)\\ W + AB & Z \end{bmatrix} < 0 \tag{15}$$

3. MAIN RESULT

Theorem 1. Given symmetric input saturation $\bar{u}_{(s)}$, the uncertain-like T-S model (6) with saturated actuators, driven by the PDC controller (8), satisfies the above defined problem statements i and ii, if there exist the matrices $\tilde{X} = \tilde{X}^T = diag(X_1, X_2) > 0$, Υ_i, Y_i, $\mathcal{N}_i$ and the scalars $\alpha > 0$ and $\gamma > 0$, such that the following optimization problem is satisfied:

$$\begin{cases} \min \gamma, \ \max \text{Trace}(X_1) \\ \text{s.t. LMIs (17), (18) and (19).} \end{cases} \tag{16}$$

$$\begin{bmatrix} \tilde{X} & (*) \\ \begin{bmatrix} Y_{i_{(s)}} - \mathcal{N}_{i_{(s)}} & 0_{1\times n} \end{bmatrix} & \bar{u}^2_{(s)} \end{bmatrix} \geq 0, \ \forall s \in \mathcal{I}_3, \ \forall i \in \mathcal{I}_4 \tag{17}$$

$$\begin{bmatrix} \tilde{X} & (*) \\ \mathfrak{L}_{(s)}\tilde{X} & \mathcal{Q}^2_{(s)} \end{bmatrix} \geq 0, \ \forall s \in \mathcal{I}_4, \tag{18}$$

$$\left[\begin{array}{cc|cc} \multicolumn{2}{c|}{\Gamma_i} & (*) & (*) \\ \hline \bar{\Upsilon}_i \tilde{B}^T + \mathcal{N}_i & 0 & -2\bar{\Upsilon}_i & (*) \\ \tilde{F}^T & 0 & 0 & -\gamma^2 I \end{array}\right] < 0, \forall i \in \mathcal{I}_4 \tag{19}$$

with $\Gamma_i = \begin{bmatrix} \Gamma_i^{11} & (*) & (*) & (*) & (*) \\ X_1(\mathcal{A}_i^T - A_r^T) & \mathcal{H}(X_2 A_r^T) & (*) & (*) & (*) \\ E_a X_1^T & 0 & -\alpha I & (*) & (*) \\ 0 & E_a X_1 & 0 & -\alpha I & (*) \\ X_1 & 0 & 0 & 0 & -I \end{bmatrix}$ and $\Gamma_i^{11} = \mathcal{H}(X_1 \mathcal{A}_i^T + Y_i^T B^T) + 2\alpha H H^T$.

In that case, the PDC controller gain and Lyapunov matrices can be respectively recovered by the changes of variables $K_i = Y_i X_1^{-1}$ ($\forall i \in \mathcal{I}_4$) and $\bar{P} = \tilde{X}^{-1}$, so that an estimate of the closed-loop domain of attraction $\mathcal{D}_a$ is given by the Lyapunov level set $\mathcal{L}(1)$, which edge is characterized by the surface $\tilde{x}^T \bar{P} \tilde{x} = 1$.

Proof. Consider the quadratic Lyapunov function candidate $V(\tilde{x}(t)) = \tilde{x}(t)^T \bar{P} \tilde{x}(t)$ with $\bar{P} = \bar{P}^T > 0$. The closed loop system (9) satisfies the above defined problem statements i and ii, and the input sector constraint is respected from Lemma 1, if:

$$\begin{aligned} &\dot{\tilde{x}}^T \bar{P} \tilde{x} + \tilde{x}^T \bar{P} \dot{\tilde{x}} + \tilde{x}^T \tilde{Q} \tilde{x} - \gamma^2 \omega^T \omega - 2\Psi^T(u)\Upsilon_h(\Psi(u) - \mathcal{T}_h(x - x_r)) \\ &= \tilde{x}^T(\bar{A}_h^T \bar{P} + \bar{P}\bar{A}_h + \tilde{Q})\tilde{x} + \tilde{x}^T \bar{P}\tilde{F}\omega + \omega^T \tilde{F}^T \bar{P}\tilde{x} + \tilde{x}^T \bar{P}\tilde{B}\Psi(u) \\ &+ \Psi(u)^T \tilde{B}^T \bar{P}\tilde{x} - \gamma^2 \omega^T \omega - 2\Psi(u)^T \Upsilon_h(\Psi(u) - \mathcal{T}_h(x - x_r)) < 0 \end{aligned} \tag{20}$$

That is to say, $\forall \begin{bmatrix} \tilde{x}^T(t) & \Psi^T(u) & \omega^T(t) \end{bmatrix} \neq 0$, if:

$$\begin{bmatrix} \tilde{A}_h^T \bar{P} + \bar{P}\tilde{A}_h + \tilde{Q} & (*) & (*) \\ \tilde{B}^T \bar{P} + \Upsilon_h \mathcal{T}_h^* & -2\Upsilon_h & (*) \\ \tilde{F}^T \bar{P} & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{21}$$

where $\mathcal{T}_h^* = [\mathcal{T}_h \ 0]$.

Let $\tilde{X} = \bar{P}^{-1}, \bar{\Upsilon}_h = \Upsilon_h^{-1}$. Multiplying (21) left by $diag(\tilde{X}, \bar{\Upsilon}_h, I)$ and right by its transpose, it yields:

$$\begin{bmatrix} \tilde{X}\bar{A}_h^T + \bar{A}_h \tilde{X}^T + \tilde{X}\tilde{Q}\tilde{X}^T & (*) & (*) \\ \bar{\Upsilon}_h \tilde{B}^T + \mathcal{N}_h & -2\bar{\Upsilon}_h & (*) \\ \tilde{F}^T & 0 & -\gamma^2 I \end{bmatrix} < 0 \tag{22}$$

with $= \mathcal{N}_h = \mathcal{T}_h^* \tilde{X}$.

Let $\Gamma_h = \tilde{X}\bar{A}_h^T + \bar{A}_h \tilde{X}^T + \tilde{X}\tilde{Q}\tilde{X}^T$, then opening the matrix $\bar{A}_h$, $\tilde{X}$ and $\tilde{Q}$, we have:

$$\Gamma_h = \begin{bmatrix} \mathcal{H}(X_1(\mathcal{A}_h^T + \Delta\mathcal{A}^T + K_h^T B^T)) + X_1 X_1 & (*) \\ X_1(\mathcal{A}_h^T - A_r^T + \Delta\mathcal{A}^T) & \mathcal{H}(X_2 A_r) \end{bmatrix} < 0 \tag{23}$$

Then, applying Lemma 2 with $M = \alpha I$, we can write:

$$\Gamma_h \leq \begin{bmatrix} \mathcal{H}(X_1(\mathcal{A}_h^T + K_h^T B^T)) + X_1 X_1 + \Omega_1 & (*) \\ X_1(\mathcal{A}_h^T - A_r^T) & \mathcal{H}(X_2 A_r^T) + \Omega_2 \end{bmatrix} < 0 \tag{24}$$

with $\Omega_1 = 2\alpha H H^T + \alpha^{-1} X_1 E_a^T E_a X_1^T$ and $\Omega_2 = \alpha^{-1} X_1 E_a^T E_a X_1^T$.
Therefore, from (24) we can major (23) then apply the Schur complement lead to the conditions expressed as LMIs (19).

Now, to provide an estimate $\mathcal{D}_a$ of the closed-loop domain of attraction, note that, from inequalities (17) respectively (18), the inclusion $\mathcal{D}_a \subseteq \mathcal{D}_u$ respectively $\mathcal{D}_a \subseteq \mathcal{D}_{\tilde{x}}$ is ensured. Indeed, without loss of generality, let us consider the Lyapunov level set $\mathcal{L}(1)$ defined, at $t = 0$, by:

$$\mathcal{L}(1) = \{\tilde{x}(0) : \tilde{x}^T(0)\bar{P}\tilde{x}(0) \leq 1\} \tag{25}$$

Then, applying the Schur complement on (17), we get:

$$\tilde{X} - \frac{\begin{bmatrix} Y_{i_{(s)}} - \mathcal{N}_{i_{(s)}} & 0_{1\times n} \end{bmatrix}^T \begin{bmatrix} Y_{i_{(s)}} - \mathcal{N}_{i_{(s)}} & 0_{1\times n} \end{bmatrix}}{\bar{u}^2_{(s)}} \geq 0 \tag{26}$$

Pre and post multiplying (26) by $\tilde{x}^T(0)\tilde{X}^{-T}$ and its transpose, it yields:

$$\tilde{x}^T(0) \frac{\begin{bmatrix} K_{i_{(s)}} - \mathcal{T}_{i_{(s)}} & 0_{1\times n} \end{bmatrix}^T \begin{bmatrix} K_{i_{(s)}} - \mathcal{T}_{i_{(s)}} & 0_{1\times n} \end{bmatrix}}{\bar{u}^2_{(s)}} \tilde{x}(0) - \tilde{x}^T(0)\bar{P}\tilde{x}(0) \leq 0 \tag{27}$$

Then, it follows: $\left| \begin{bmatrix} K_{i_{(s)}} - \mathcal{T}_{i_{(s)}} & 0_{1\times n} \end{bmatrix} \tilde{x}(0) \right| \leq \bar{u}^2_{(s)}$, so, from Lemma 1, all initial condition $\tilde{x}(0) \in \mathcal{L}(1) \subseteq \mathcal{D}_u$.

Moreover, from (18) and the Schur complement we get:

$$\frac{\tilde{X}^T \mathfrak{L}^T_{(s)} \mathfrak{L}_{(s)} \tilde{X}}{\mathcal{Q}^2_{(s)}} - \tilde{X} \leq 0 \tag{28}$$

Similarly, after congruence by $\tilde{X}^{-1}\tilde{x}(0)$, this provides $|\mathfrak{L}_{(s)}\tilde{x}(0)| \leq \mathcal{Q}_{(s)}$, so all initial conditions $\tilde{x}(0) \in \mathcal{L}(1) \subseteq \mathcal{D}_{\tilde{x}}$.

Consequently $\mathcal{D}_a = \mathcal{L}(1) \subseteq \mathcal{D}_{\tilde{x}} \cap \mathcal{D}_u$. Finally, a simple procedure to enlarge $\mathcal{L}(1)$ is to minimize the trace of $\bar{P}$. However, since we are mainly concerned to enlarge the domain of attraction with regards to the tracking error $e(t) = x(t) - x_r(t)$, we can restrict such optimization procedure by only maximizing the trace of X_1. □

4. SIMULATION RESULTS

In the previous section, LMI conditions for model reference-based tracking PDC controller designed for a class of uncertain-like T-S systems under actuators saturation have been proposed. Simulation results are presented in this section to illustrate their effectiveness. To do so, we assume a linear reference model (7) (with A_r Hurwitz) specified by:

$$A_r = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -6 & 0 & 0 & -4 & 0 & 0 \\ 0 & -6 & 0 & 0 & -4 & 0 \\ 0 & 0 & -6 & 0 & 0 & -4 \end{bmatrix}, \ B_r = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.05 \\ 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

The bounds for the actuators saturation are computed from (3), assuming $\varpi_j(t) \in [0, 100]$ (%, $j \in \mathcal{I}_4$), such that $\bar{u}_1 = 0.102$, $\bar{u}_2 = 0.102$, $\bar{u}_3 = 0.104$, $\bar{u}_g = (4\varpi_{max})\kappa_a + \kappa_b = 1615.8$. Also, we assume $\bar{x}_4 = \bar{x}_5 = \pi$, which define $\mathcal{D}_x$ (see (10)). The LMI conditions presented in Theorem 1 are solved using Yalmip in MATLAB with SeDuMi Solver (Lofberg, 2004). We obtained the minimized H_∞ attenuation level $\gamma = 0.71$ with the following gain matrices of the PDC tracking controller (8) and Lyapunov matrices ($\bar{P} = diag(\bar{P}_1, \bar{P}_2)$).

$$K_1 = \begin{bmatrix} -0.8636 & -0.0006 & -0.0023 & -0.7271 & -0.0005 & -0.0017 \\ 0.0075 & -1.1589 & 0.0049 & 0.0066 & -0.9828 & 0.0035 \\ 0.0496 & -0.0833 & -0.3373 & 0.0436 & -0.0739 & -0.2400 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} -0.8636 & 0.0006 & 0.0023 & -0.7271 & 0.0005 & 0.0017 \\ -0.0075 & -1.1589 & 0.0049 & -0.0066 & -0.9828 & 0.0035 \\ -0.0496 & -0.0833 & -0.3373 & -0.0436 & -0.0739 & -0.2400 \end{bmatrix}$$

$$K_3 = \begin{bmatrix} -0.8636 & 0.0006 & -0.0023 & -0.7271 & 0.0005 & -0.0017 \\ -0.0075 & -1.1589 & -0.0049 & -0.0066 & -0.9828 & -0.0035 \\ 0.0496 & 0.0833 & -0.3373 & 0.0436 & 0.0739 & -0.2400 \end{bmatrix}$$

$$K_4 = \begin{bmatrix} -0.8636 & -0.0006 & 0.0023 & -0.7271 & -0.0005 & 0.0017 \\ 0.0075 & -1.1589 & -0.0049 & 0.0066 & -0.9828 & -0.0035 \\ -0.0496 & 0.0833 & -0.3373 & -0.0436 & 0.0739 & -0.2400 \end{bmatrix}$$

$$\bar{P}_1 = \begin{bmatrix} 2.1900 & -0.0000 & 0.0000 & 0.2063 & 0.0000 & 0.0000 \\ -0.0000 & 2.1883 & 0.0000 & 0.0000 & 0.2074 & 0.0000 \\ 0.0000 & 0.0000 & 1.5130 & -0.0000 & -0.0000 & 0.0175 \\ 0.2063 & 0.0000 & -0.0000 & 0.1822 & 0.0000 & 0.0000 \\ 0.0000 & 0.2074 & -0.0000 & 0.0000 & 0.1842 & 0.0000 \\ 0.0000 & 0.0000 & 0.0175 & 0.0000 & 0.0000 & 0.0118 \end{bmatrix}$$

$$\bar{P}_2 = \begin{bmatrix} 0.5701 & -0.0000 & -0.0000 & 0.0062 & 0.0000 & -0.0000 \\ -0.0000 & 0.5690 & 0.0000 & 0.0000 & 0.0063 & 0.0000 \\ -0.0000 & 0.0000 & 0.0088 & 0.0000 & 0.0000 & 0.0041 \\ 0.0062 & 0.0000 & 0.0000 & 0.1017 & 0.0000 & 0.0000 \\ 0.0000 & 0.0063 & 0.0000 & 0.0000 & 0.1017 & 0.0000 \\ -0.0000 & 0.0000 & 0.0041 & 0.0000 & 0.0000 & 0.0044 \end{bmatrix}$$

Figures 2-4 illustrate the effectiveness of the proposed tracking control methodology under the input constraints. For that simulation, the initial conditions are set to $x(0) = \left[\frac{-\pi}{3}\ \frac{\pi}{3}\ \frac{\pi}{3}\ \frac{\pi}{3}\ \frac{\pi}{3}\ \frac{\pi}{3}\right]^T$ for the nonlinear model of the quadrotor (1) and $x_r(0) = [0\,0\,0\,0\,0\,0]^T$ for the reference model (7). The disturbance and reference signals are respectively set, for all $t \in [0, 10s]$, to $\varphi(t) = \sin(\pi t + 0.5)$ and $r(t) = \left[1.4\sin(0.6\pi t)\ \ 1.5\sin(0.28\pi t))\ \ 0.8\,\mathrm{square}(0.2\pi t)\right]^T$, and 0 otherwise.

Fig. 2 exhibits the angular positions, roll $\phi(t) = x_1(t)$, pitch $\theta(t) = x_2(t)$ and yaw $\psi(t) = x_3(t)$ of the quadrotor with respect to the reference signals denoted as $x_{1,r}(t)$, $x_{2,r}(t)$ and $x_{3,r}(t)$, respectively. Fig. 3 shows the angular velocities $x_4(t)$, $x_5(t)$, $x_6(t)$ of the quadrotor following the reference signals $x_{4,r}(t), x_{5,r}(t)$ and $x_{6,r}(t)$. The control inputs of the quadrotors, namely $u_1(t)$, $u_2(t)$ and $u_3(t)$ are depicted in Fig. 4 showing saturation during the state errors' transients, the saturated inputs being fed to the quadrotor. These figures show that the proposed tracking PDC controller design for the considered quadrotor nonlinear model, which states track the reference model's ones, provides good results despite the disturbance signals and input constraints, then stabilizes when $\omega(t)$ gets to 0.

Finally, because of the input constraints, recall that these results only hold regionally. Of course, because we are dealing with a twelve order system, it is not possible to plot the whole estimate of the domain of attraction $\mathcal{D}_a = \mathcal{L}(1)$ on a 3D graphic. However, to illustrate such estimation, Fig. 5 shows projections of $\mathcal{L}(1)$ on some planes of interest, respectively (x_4, x_5), $(x_{4,r}, x_{5,r})$ and (e_4, e_5). Moreover, in this figure, a 3D graphic of $u_s^2 \in [0, 0.012]$ regarding to (e_4, e_5) to illustrate the constrained input domain $\mathcal{D}_u$. It is worth noticing that the extended state trajectories (plotted as blue lines) stays inside $\mathcal{L}(1)$, as expected. This also confirms the effectiveness of our proposal.

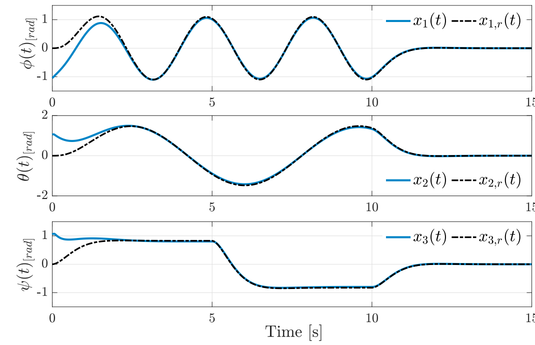

Fig. 2. Reference vs Quadrotor's angular positions.

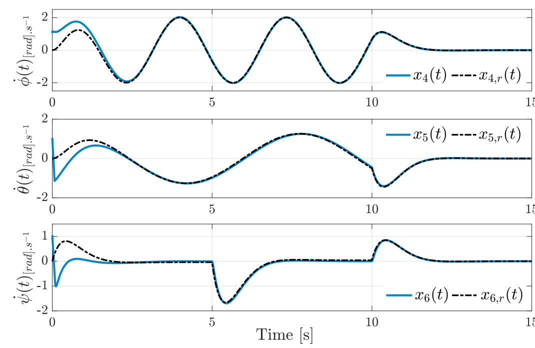

Fig. 3. Reference vs quadrotor's angular velocities.

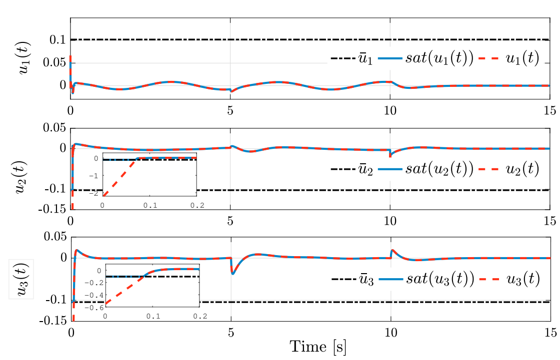

Fig. 4. Control signals.

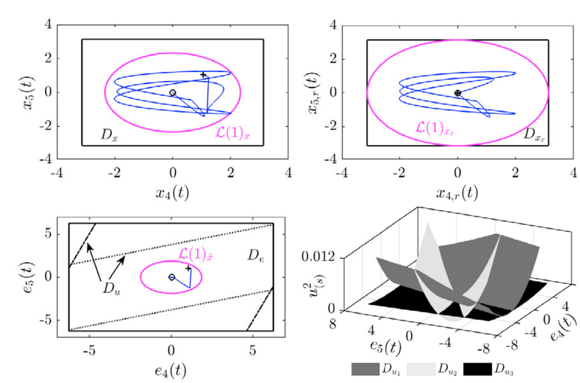

Fig. 5. Projection of $\mathcal{D}_a^* = \mathcal{L}(1)$ (pink lines), $\mathcal{D}_x$ (black lines), $\mathcal{D}_{\tilde{x}} = \mathcal{D}_e \cup \mathcal{D}_{x_r}$ (black lines) and $\mathcal{D}_u$ (black doted lines) on 2D planes and 3D space of interest with closed-loop trajectories in $\mathcal{L}(1)$ (blue lines).

5. CONCLUSION

In this paper, the regional attitude tracking control of a quadrotor UAV under actuator saturation's has been investigated. The dynamical model of the quadrotor's attitude has been described as an uncertain-like T-S model. The input saturation of the quadrotors has been dealt with the generalized sector conditions. Then, LMI conditions were derived for the design of the model reference tracking controller using Lyapunov functions, along with the L_2 norms minimisation of the transfer between the disturbances and the tracking errors. Simulation results were provided to illustrate the effectiveness of the proposed control scheme as well as investigate the closed-loop domains of attraction, with regards to the input constraints. In our further works the full control of the quadrotor, including the tracking in its xyz positions, will be investigated in order to proceed with aggressive maneuvers tracking.

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